Diploma Programme
Programme du diplôme
Programa del Diploma

Further mathematics
Higher level
Paper 2

Friday 24 May 2019 (morning)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

8 pages

2219 – 7102

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Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 12]

A large group of students sit two Mathematics examinations, P1 and P2. Mr Brown, their teacher, wishes to carry out an early statistical analysis of the results so he selects 10 students at random and he marks both papers submitted by these students. He obtains the following results. You may assume that the pairs of marks follow a bivariate normal distribution.

P1	50	62	51	77	72	43	83	69	72	53
P2	56	49	52	72	71	45	78	68	65	51

(a) Mr Brown decides to carry out a test to investigate whether or not the mean marks on the two papers will turn out to be equal. He defines the following hypotheses

$$H_0: d = 0; H_1: d \neq 0$$

where d denotes the difference in the mean marks on the two papers.

(i) Carry out an appropriate test, stating the degrees of freedom, the value of the test statistic and the p-value.

(ii) State your conclusion in context. [7]

(b) Mr Brown now wishes to investigate whether or not there is a positive association between the marks of the students on the two papers.

(i) State suitable hypotheses.

(ii) Determine the value of r, the product moment correlation coefficient of these data.

(iii) State the p-value and interpret it in context. [5]

2. [Maximum mark: 20]

(a) By considering the images of $(1, 0)$ and $(0, 1)$, show that the matrix associated with an anticlockwise rotation of points through an angle θ about the origin is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$ [2]

(b) The conic C_1 has equation

$$x^2 + y^2 - 6xy - 4 = 0.$$

C_1 is rotated 45° anticlockwise about the origin to form the conic C_2.

(i) Show that the equation of C_2 is

$$x^2 - \frac{y^2}{2} = 1.$$

(ii) Identify C_2 as a parabola, an ellipse or a hyperbola. [7]

(c) For C_2 determine the

(i) eccentricity;

(ii) coordinates of the foci;

(iii) equations of the directrices. [7]

(d) Hence, for C_1, determine the

(i) coordinates of the foci;

(ii) equations of the directrices. [4]

3. [Maximum mark: 17]

The matrix $\boldsymbol{M}$ is given by $\boldsymbol{M} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$.

(a) Given that $\boldsymbol{M}^3$ can be written as a quadratic expression in $\boldsymbol{M}$ in the form $a\boldsymbol{M}^2 + b\boldsymbol{M} + c\boldsymbol{I}$, determine the values of the constants a, b and c. [7]

(b) Show that $\boldsymbol{M}^4 = 19\boldsymbol{M}^2 + 40\boldsymbol{M} + 30\boldsymbol{I}$. [2]

(c) Using mathematical induction, prove that $\boldsymbol{M}^n$ can be written as a quadratic expression in $\boldsymbol{M}$ for all positive integers $n \geq 3$. [6]

(d) Find a quadratic expression in $\boldsymbol{M}$ for the inverse matrix $\boldsymbol{M}^{-1}$. [2]

4. [Maximum mark: 12]

(a) Consider the congruence $7^{22} \equiv 1 \pmod{23}$.

(i) Name and state the theorem which justifies this congruence.

(ii) Use the binomial theorem to prove that $7^{22n} \equiv 1 \pmod{23}$ where $n \in \mathbb{Z}^+$.

(iii) Find the value of $7^{2209} \pmod{23}$ lying in the interval $[1, 22]$. [7]

(b) (i) Find the smallest positive integer a satisfying $7^a \equiv 1 \pmod{22}$.

(ii) Given that $7^b \equiv 9 \pmod{22}$ find all possible values of b. [5]

5. [Maximum mark: 21]

(a) Consider the set $G = \{1, 2, 3, 4, 5, 6\}$ and the operation $\times_7$ which denotes multiplication modulo 7.

(i) Copy and complete the following Cayley table for G.

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	
3	3	6	2	5		
4	4	1				
5	5					
6	6					

(ii) Show that $\{G, \times_7\}$ is a group.

(iii) Show that $\{G, \times_7\}$ is cyclic. [9]

(b) The groups A and B are both cyclic groups of order n with generators a and b respectively.
It is given that the mapping from A to B defined by $a^p \to b^p$, where $1 \le p \le n$, is a bijection. Use this bijection to show that A and B are isomorphic. [6]

The set $H = \{0, 1, 2, 3, 4, 5\}$ and $+_6$ denotes addition modulo 6. You are given that $\{H, +_6\}$ is a cyclic group and therefore isomorphic to $\{G, \times_7\}$.

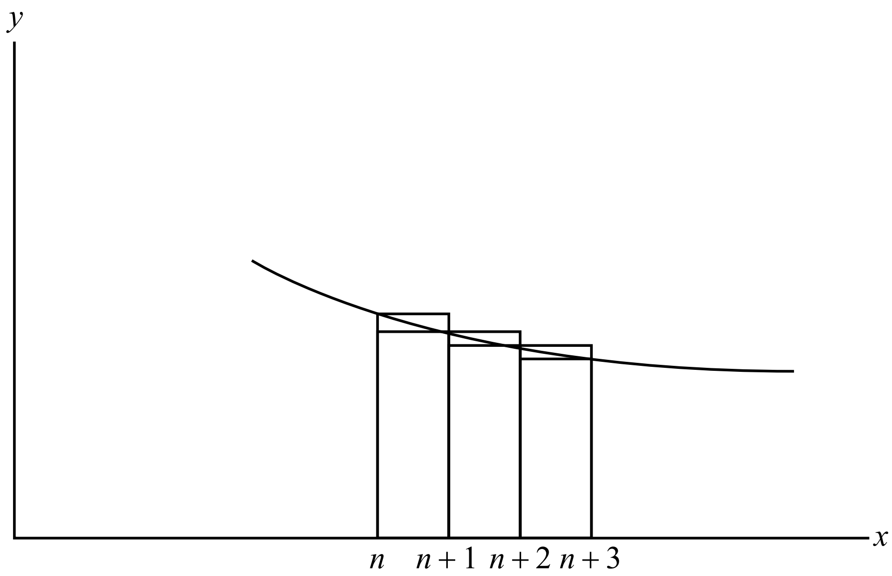
(c) Find two distinct bijections from G to H which could be used to define the isomorphism. For each bijection, you should give the element in H to which each element in G is mapped. [6]

6. [Maximum mark: 15]

(a) Using the substitution $u = x^3$, show that

$$\int \frac{x^2}{x^6+1}\,\mathrm{d}x = \frac{1}{3}\arctan(x^3) + \text{constant}\,.$$ [2]

(b) The following diagram shows a sketch of part of the curve $y = \frac{x^2}{x^6+1}$ together with line segments parallel to the coordinate axes. The vertical line segments are at $x = n$, $x = n+1$, $x = n+2$, $x = n+3$ where $n \in \mathbb{Z}^+$.



(i) By considering upper and lower Riemann sums, show that

$$\sum_{r=n+1}^{\infty}\frac{r^2}{r^6+1} < \int_n^{\infty}\frac{x^2\,\mathrm{d}x}{x^6+1} < \sum_{r=n}^{\infty}\frac{r^2}{r^6+1}\,.$$

(ii) Let $S = \sum_{r=1}^{\infty}\frac{r^2}{r^6+1}$. Using the result in (i), show that an upper bound for S is given by

$$\sum_{r=1}^{n}\frac{r^2}{r^6+1} + \frac{\pi}{6} - \frac{1}{3}\arctan(n^3)\,.$$

(iii) Find a similar result which gives a lower bound for S.

(iv) Hence, by taking $n = 8$, determine the value of S correct to as many decimal places as your upper and lower bounds allow. [13]

7. [Maximum mark: 22]

The discrete random variable X has the following probability distribution, where θ is a constant.

x	0	1	2
$\mathrm{P}(X=x)$	θ	2θ	$1-3\theta$

(a) State the range of possible values of θ. [2]

(b) Show that $\mathrm{E}(X^n)=2^n+2\theta(1-3\times 2^{n-1})$. [2]

In order to estimate θ, a random sample $X_1, X_2, \ldots, X_n$ is taken from the distribution of X.

(c) Show that $T_1=\frac{1}{2}-\frac{1}{4n}\sum_{i=1}^{n}X_i$ is an unbiased estimator for θ. [3]

(d) Show that $T_2=\frac{2}{5}-\frac{1}{10n}\sum_{i=1}^{n}X_i^2$ is also an unbiased estimator for θ. [2]

(e) (i) Show that $\mathrm{Var}(X)=6\theta-16\theta^2$.

(ii) Express $\mathrm{Var}(X^2)$ in the form $a\theta-b\theta^2$, where $a, b\in\mathbb{Z}^+$. [6]

(f) Determine which is the more efficient estimator, T_1 or T_2. [7]

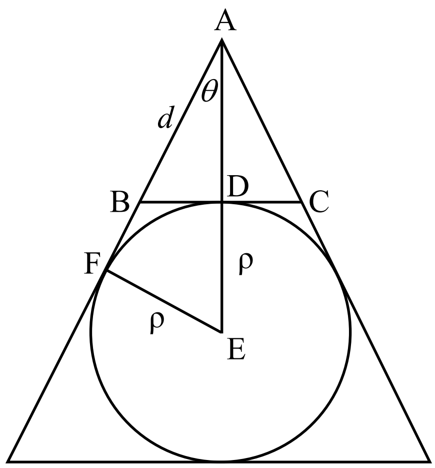
8. [Maximum mark: 16]

The triangle T is isosceles with $\mathrm{AB} = \mathrm{AC} = d$ and $\mathrm{B\hat{A}C} = 2\theta$. In parts (a) and (b) you are required to draw a diagram to illustrate the situation.

(a) Find an expression for the radius R of the circumscribed circle of T in terms of d and θ. [4]

(b) Show that the radius r of the inscribed circle of T is equal to $\dfrac{d\sin\theta\cos\theta}{1+\sin\theta}$. [4]

The following diagram illustrates another circle which is outside the triangle T.
It is tangential to $[\mathrm{BC}]$, and also to the other two sides $[\mathrm{AB}]$ and $[\mathrm{AC}]$ which have been extended. This is called the escribed circle of T which has radius ρ, centre E and points of tangency D and F.



(c) Find an expression for the radius ρ in terms of d and θ. [4]

(d) Show that, for an equilateral triangle, the radii of the inscribed circle, the circumscribed circle and the escribed circle form an arithmetic sequence. [4]

9. [Maximum mark: 15]

Consider the second-degree recurrence relation $u_n = 4u_{n-1} - 5u_{n-2}$.

(a) (i) Write down and solve the auxiliary equation.

(ii) Hence write down the general solution to this recurrence relation.

(iii) Given the initial conditions $u_0 = u_1 = 2$, show that

$u_n = (1 + \mathrm{i})(2 + \mathrm{i})^n + (1 - \mathrm{i})(2 - \mathrm{i})^n$ for all $n \in \mathbb{N}$. [8]

(b) Show that an alternative expression for u_n is

$u_n = 2^{\frac{3}{2}}5^{\frac{n}{2}}\cos\left(\dfrac{\pi}{4} + n\arctan\left(\dfrac{1}{2}\right)\right)$ for all $n \in \mathbb{N}$. [7]