INTRODUCTION

-This book is written mainly for Zambian pupils in Grade 10 – 12 pupils, but it can also be useful to anyone seeking a basic knowledge of Mathematics.
-It is intended for both study and revision.
-It covers the Mathematics High School Syllabus in Zambia as much as possible, but the contents of this book are not enough to get 100% of knowledge in Mathematics.
-Teacher's explanation is needed to learn the contents in this book even if all pupils have this textbook.
-This book is sold through JOCV (Japan Overseas Cooperation Volunteers) teacher's committee only.
-The price can be decided as the lowest price by only JOCV teacher's committee.
-The price includes only the expense for the materials needed to produce this booklet and no one will make profit from this booklet.
-Only JOCV can edit this book.

ACKNOWLEDGEMENT

I would like to say thanks to all mathematicians, authors and publishers, especially to J F Talbert 'Success in Examinations Mathematics Syllabus D', Karen Morisson 'IGCSE Mathematics', Tori Large 'Usborne Dictionary of Math', Frank Tapson 'Barron's Mathematics Study Dictionary' and Hidenori KUGA 'Masamu Ofunica' for giving many useful ideas.
Thanks Peter Kausa and Patience Kabaso for giving the wonderful title and book cover.

I hope that this book helps all Zambian pupils make progress in Mathematics.

March 2009
Motoki KAMIZONO from DENSO
2007 – 2009 Nyimba Boarding Secondary School
Japan Overseas Cooperation Volunteers - JOCV

simsiky@gmail.com

CONTENTS

1 Basic Arithmetic

(ア) Numbers

Integers: are numbers which are not fractions. . . . –4, –3, –2, –1, 0, 1, 2, 3, 4, . . .

The set of integers is denoted by the symbol **Z**. ← negative] [positive →

Natural numbers: are the positive integers. 1, 2, 3, 4, 5, . . .

The set of natural numbers is denoted by the symbol **N**.

Whole numbers: are the set of natural numbers and zero.

Even numbers: are integers which can be divided by 2 without a remainder.

Any integer which ends with 0, 2, 4, 6 or 8 must be even.

Odd numbers: are integers which cannot be divided by 2 without a remainder.

Any integer which ends with 1, 3, 5, 7 or 9 must be odd.

Prime numbers: are numbers which can only be divided by 1 and themselves.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, . . .

There is an infinite number of prime numbers.

・1 is NOT a prime number.

・2 is the only even prime number.

Rational numbers: are numbers which can be written in the form $\frac{a}{b}$ where *a* and *b* are integers and *b* is not zero.

e.g. $-2.5,\ 0.\overline{3},\ 0.\overline{09},\ 3.\overline{142857}$ and 4 are all rational since they can be re-written as: $-\frac{5}{2},\ \frac{1}{3},\ \frac{1}{11},\ \frac{22}{7}$ and $\frac{4}{1}$ respectively.

Irrational numbers: are numbers which are not rational and so cannot be written in the form $\frac{a}{b}$.

e.g. $\sqrt{2} = 1.41421356\ldots$, $\pi = 3.141592653\ldots$

Real numbers: The set of real numbers is made up of all rational and irrational numbers together.

The set of real numbers is denoted by the symbol **R**.

Significant figures (sf): indicate the accuracy of numbers.

For any number, the first significant figure is the first non-zero digit. After that,

- in case of whole numbers, all digits are counted as significant up to the last non-zero digit.

- in case of decimals, all digits are counted as significant.

e.g.

4 sf	1234	12.34	1.023	0.01234	0.001002
3 sf	1230	12.3	1.02	0.0123	0.00100
2 sf	1200	12	1.0	0.012	0.0010
1 sf	1000	10	1	0.01	0.001

(イ) Operations on Real Numbers

Order of operation s: There is an established order in which operations must be done. Anything in brackets has to be done first, then multiplication and division, and then addition and subtraction. An aid to memorise this order is BoDMAS (Brackets of Division, Multiplication, Addition and Subtraction)

e.g. $(2+4)\times 3-1 = 6\times 3-1 = 18-1 = 17$

$2+4\times(3-1) = 2+4\times 2 = 2+8 = 10$

$(2+4)\times(3-1) = 6\times 2 = 12$

$2+4\times 3-1 = 2+12-1 = 13$

The commutative law: states that for any two real numbers, the order of operation of addition and multiplication doesn't matter.

$\boxed{a+b=b+a,\ ab=ba}$ e.g. $3+5=5+3,\ 3\times 6=6\times 3$

BUT subtraction and division are not commutative.

$a-b\neq b-a,\ a\div b\neq b\div a$ e.g. $3-5\neq 5-3,\ 3\div 6\neq 6\div 3$

The associative law: states that for any three real numbers, the grouping in an expression of addition and multiplication doesn't matter.

$\boxed{(a+b)+c=a+(b+c),\ (ab)c=a(bc)}$ e.g. $(2+3)+4=2+(3+4),\ (2\times 3)\times 4=2\times$

BUT subtraction and division are not associative.

$(a-b)-c\neq a-(b-c)$ e.g. $(2-3)-4\neq 2-(3-4)$

$(a\div b)\div c\neq a\div(b\div c)$ e.g. $(2\div 3)\div 4\neq 2\div(3\div 4)$

simsiky@gmail.com

The distributive law: states that when it is performed on two or more quantities already combined by another operation, the result is the same as when it is performed on each quantity individually and the products then combined.

$a(b+c)=ab+ac$ e.g. $2\times(3+4)=2\times3+2\times4$

（ウ）Factors and Multiples

Factors of a number: A factor is a whole number which divides exactly into it.
1 is a factor of every number and every number is a factor of itself.
e.g. The factors of 12 are 1, 2, 3, 4, 6 and 12.

Any whole number can be written as a product of its factors.
e.g. 12 = 1 × 12, 12 = 2 × 6, 12 = 3 × 4

Prime factors: The prime factors of a number are factors which are also prime numbers.
e.g. The prime factors of 12 are 2 and 3.

A non-prime number can be expressed uniquely as a product of prime factors.
e.g. $60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$

> To find the factors of a number, divide by the prime numbers in turn until no further division is possible.

Example 1 Express 360 as the product of its prime factors.

2	360
2	180
2	90
3	45
3	15
	5

$\therefore 360 = 2\times2\times2\times3\times3\times5$
$= 2^3\times3^2\times5$

Highest Common Factor (HCF): The HCF of two (or more) numbers is the largest number which is a factor of each.

e.g. 12 has 1, 2, 3, 4, 6 and 12 as factors.
18 has 1, 2, 3, 6, 9 and 18 as factors.
The common factors of 12 and 18 are 1, 2, 3 and 6.
The HCF is 6.

> To find the HCF, express each number as a product of the prime factors and select all the common factors.

Example 2 Find the HCF of 168, 252 and 360.

168 = **2**×**2**×2×**3**×7
252 = **2**×**2**×**3**×3×7
360 = **2**×**2**×2×**3**×3×5

2^2 and 3 are the only common factors.

Therefore the HCF = 2 × 2 × 3 = 12

Multiples of a number: A multiple is a number made by multiplying that number with a whole number.
e.g. The multiples of 3 are 0, 3, 6, 9, 12, . . .

Lowest Common Multiple (LCM): The LCM of two (or more) numbers is the smallest number which is a multiple of each.

> To find the LCM, express each number as a product of the prime factors and select all the different factors without repetition.

Example 3 Find the LCM of 6, 12 and 15

6 = 2 × 3
12 = 2 × 2 × 3
15 = 3 × 5
Therefore the LCM = 2 × 2 × 3 × 5 = 60

（エ）Conversions of Fractions, Decimals and Percentages

Converting a fraction to a decimal: divide the numerator by the denominator.

e.g. $\frac{1}{8}=0.125 \quad \because 8\overline{)1.000}$ = 0.125

Converting a decimal to a fraction: place the decimal numbers over the appropriate number of power of 10.

e.g. $0.3=\frac{3}{10}$, $4.25=4\frac{25}{100}=4\frac{1}{4}$

Converting a fraction to a percentage: multiply the fraction by 100 and put %.

e.g. $\frac{2}{5}=\frac{2}{5}\times100\%=40\%$, $\frac{1}{8}=\frac{1}{8}\times100\%=12\frac{1}{2}\%$ or 12.5%

Converting a percentage to a fraction: divide the percentage by 100 and remove %.

e.g. $5\% = \frac{5}{100} = \frac{1}{20}$, $12.5\% = \frac{12.5}{100} = \frac{12.5}{100} \times \frac{10}{10} = \frac{125}{1000} = \frac{1}{8}$

Converting a decimal to a percentage: shift the decimal point two places to the RIGHT.

e.g. 0.25 = 25%, 0.0075 = 0.75%, 1.5 = 150%

Converting a percentage to a decimal: shift the decimal point two places to the LEFT.

e.g. 78.5% = 0.785, 120% = 1.2

(オ) Operations of Fractions

Adding and subtracting fractions: express each fraction in terms of the same denominator (the LCM) and add / subtract the numerators.

Example 4 Calculate $\frac{1}{3} + \frac{2}{5} - \frac{1}{4}$

LCM of 3, 5 ,4 is 60. $\frac{1}{3} + \frac{2}{5} - \frac{1}{4} = \frac{20 + 24 - 15}{60} = \frac{29}{60}$

With mixed numbers, either convert to improper fractions first or deal with whole numbers separately.

Example 5 Calculate $1\frac{1}{5} - 2\frac{3}{4} + 3\frac{1}{2}$

$$1\frac{1}{5} - 2\frac{3}{4} + 3\frac{1}{2} = \frac{6}{5} - \frac{11}{4} + \frac{7}{2} = \frac{24 - 55 + 70}{20} = \frac{39}{20}\left(= 1\frac{19}{20}\right)$$

$$\text{or } = 1 - 2 + 3 + \frac{1}{5} - \frac{3}{4} + \frac{1}{2} = 2 + \frac{4 - 15 + 10}{20} = 2 - \frac{1}{20} = 1\frac{19}{20}$$

Multiplying fractions: multiply the numerators together and then denominators together (after any cancelling which may be possible).

Example 6 Calculate $\frac{3}{4} \times \frac{1}{2} \times \frac{5}{6}$

$$\frac{3}{4} \times \frac{1}{2} \times \frac{5}{6} = \frac{\overset{1}{\cancel{3}} \times 1 \times 5}{4 \times 2 \times \underset{2}{\cancel{6}}} = \frac{1 \times 1 \times 5}{4 \times 2 \times 2} = \frac{5}{16}$$

To multiply mixed numbers, first convert them to improper fractions.

Example 7 Calculate $6 \times 1\frac{2}{3} \times \frac{2}{5}$

$$6 \times 1\frac{2}{3} \times \frac{2}{5} = \frac{6}{1} \times \frac{5}{3} \times \frac{2}{5} = \frac{\overset{2}{\cancel{6}} \times \overset{1}{\cancel{5}} \times 2}{1 \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{5}}} = \frac{2 \times 1 \times 2}{1 \times 1 \times 1} = 4$$

Dividing fractions: To divide by a fraction, multiply by its reciprocal. (The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.)

Example 8 Calculate $\frac{3}{5} \div \frac{2}{3}$

$$\frac{3}{5} \div \frac{2}{3} = \frac{3}{5} \times \frac{3}{2} = \frac{3 \times 3}{5 \times 2} = \frac{9}{10}$$

(カ) Ratio and Scale

Ratio: is used to compare the sizes of two (or more) quantities. Ratios are written with a colon (:).

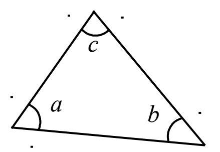
e.g. If there are 25 girls and 23 boys in a class, the ratio of girls to boys is said to be 25 : 23 (this can also be written as the fraction $\frac{25}{23}$) and the ratio of boys to girls is said to be 23 : 25.

Example 9 If angles a, b and c in a triangle are in the ratio 4 : 3 : 5, what is the size of each angle?

Total number of parts = 4 + 3 + 5 = 12

The sum of interior angles in a triangle = 180˚

One part is $\frac{180°}{12} = 15°$



Angle a is 4 × 15˚ = 60˚ Angle b is 3 × 15˚ = 45˚ Angle c is 5 × 15˚ = 75˚

simsiky@gmail.com

Simplifying ratios: Ratios can often be simplified. To simplify a ratio, divide or multiply both parts by the same number.

Example 10 Find in simplest form the ratio of: (a) $\frac{1}{3}:2$ (b) 30min : 2hr

(a) Multiply both parts by 3 to convert the fraction into an integer.

$\frac{1}{3}\times 3=1$ and $2\times 3=6$ So $\frac{1}{3}:2=1:6$

(b) Express both parts in the same unit and then divide both parts by their HCF.

30min : 2hr = 30min : 120min (∵2hr = 2 × 60min = 120min)

= 1 : 4 (∵HCF = 30)

Scales on maps and plans: These are stated in a ratio form 1 : n .This ratio gives the representative fraction $\left(\frac{1}{n}\right)$ of the map.

e.g. A scale of 1 : 50 000 means that 1cm on the map represents 50 000cm (0.5km) on the ground.

Example 11 A map is drawn to a scale of 1 : 50 000.

(a) On the map the length of a straight road is measured as 7.5cm. What is the actual distance of the road in km?

(b) If the area of a maize field is $2km^2$, what is its area on the map in cm^2?

(a) 1cm on the map represents 50 000cm = 0.5km on the ground.

If x represents the actual distance of the road,

1 : 0.5 = 7.5 : x ∴x = 0.5 × 7.5 = 1.5 km

(b) $1cm^2$ on the map represents 0.5 × 0.5 = 0.25 km^2 on the ground.

So if y represents the area of the maize field on the map,

$1:0.25=y:2 \quad \therefore\ y=\frac{1\times 2}{0.25}=8\,\text{cm}^2$

Exercise 1

1 Find the number of significant figures. (a) 235 (b) 0.423 (c) 0.004 (d) 24 000 (e) 2.000 (f) 0.0510

2 Calculate the following.

(a) $7-2+3\times 4$ (b) $(-4)\times(-3)+(-3)\times 2$ (c) $-5\times 7-24\div 3$

3 Evaluate the following.

(a) $2\frac{3}{4}+1\frac{1}{2}-4\frac{1}{8}$ (b) $2\frac{1}{2}-4\frac{1}{6}+3\frac{2}{5}$ (c) $1\frac{1}{3}\div 1\frac{1}{6}$ (d) $2\frac{1}{3}\div\left(2\frac{1}{2}-1\frac{4}{5}\right)-1\frac{3}{4}$

4 Find the prime factors of 42, 70 and 105, and state their HCF and LCM.

5 The ratio of the numbers of boys to girls in a school is 7 : 6. If there are 455 pupils, how many boys are there?

6 Find in simplest form the ratio of: (a) 2hr 15min : 13min 30sec (b) $\frac{2}{3}:1\frac{1}{4}:1$

7 A map is drawn to a scale of 1 : 20 000.

(a) The distance of an airport runway is represented by 8cm on the map. Calculate its actual distance in km.

(b) The actual distance of a railway is 6km. Calculate the distance on the map which it represents it in cm.

(c) The area of a lake on the map is represented by $200cm^2$. Calculate the actual area of it in km^2.

(d) The actual area of the airport is $3km^2$. Calculate its area on the map in cm^2.

2 Commercial Arithmetic

（ア） Profit and Loss

Profit / Loss = Income – Cost If the answer is positive, it's a profit.
If the answer is negative, it's a loss.

Example 1 A man bought 100 loaves of bread at K3 000 each. He sold half of them at K3 500 each and the rest at K2 700 each. Find his profit or loss.

Cost = 100 × 3 000 = K300 000
Income = 50 × 3 500 + 50 × 2 700 = K310 000
Profit / Loss = 310 000 – 300 000 = K10 000 profit

（イ） Discount

A discount is a reduction in price.

Example 2 Calculate how much is paid for a television set costing K800 000 if 7% discount is allowed.

Discount = $\frac{7}{100} \times 800\ 000 = \text{K}56\ 000$

Customer pays 800 000 – 56 000 = K744 000

（ウ） Simple Interest

Interest is money paid for borrowing a sum of money which is called the **principal**. It is expressed as a percentage rate per annum.

$$\boxed{I = \frac{PRT}{100}}$$

where I is the simple interest, P is the principal, R is the rate of interest (as a percentage) and T is the time (in years). When interest is added to the principal, the sum is called the **amount**. The amount $A = P + I$

Example 3 If a person invests K5 000 000 at the rate of 8% per annum for 2 year.
Calculate: (a) the simple interest (b) the amount

(a) $I = \frac{PRT}{100} = \frac{5\ 000\ 000 \times 8 \times 2}{100} = \text{K}800\ 000$

(b) $A = P + I$ = 5 000 000 + 800 000 = K5 800 000

Example 4 K250 000 earns K10 000 interest in 6 months. Calculate the interest rate.

$$6 \text{ months} = \frac{6}{12} \text{ year} = \frac{1}{2} \text{ year}$$

$$\therefore R = \frac{I \times 100}{PT} = \frac{10\ 000 \times 100}{250\ 000 \times \frac{1}{2}} = 8\%$$

Exercise 2

1 A bookseller bought 10 copies of a book at K45 000 each. He sold 9 copies at K52 000 per copy and the rest at K40 000 per copy. Find the profit or loss.

2 During a sale, a shop gives 5% discount.
(a) What is the new price of an article priced at K120 000?
(b) What is the original price of an article sold for K28 500?

3 A man borrows K1 200 000 at 9.5% for 8 months. How much does he have to repay?

4 If K832 000 is the total value of an investment at the end of 4 years which has a simple interest rate of 7% per annum, calculate the principal.

5 Find the interest rate if the simple interest on K250 000 invested for 6 years is K120 000.

simsiky@gmail.com

3 Indices, Standard Form and Approximations

(ア) Indices

Indices indicate how many times a number is multiplied by itself.

e.g. $2 \times 2 \times 2 = 2^3$

The 3 is the power or index of the base 2.

Laws of indices: for any real number *a*,

(a) $a^1 = a$ Any number to the power 1 is itself. e.g. $5^1 = 5$

(b) $a^0 = 1$ Any number to the power 0 is 1. e.g. $10^0 = 1$

(c) $a^{-n} = \frac{1}{a^n}$ A negative power indicates the reciprocal of the number with a positive power. e.g. $10^{-2} = \frac{1}{10^2}\left(= \frac{1}{100} = 0.01\right)$

Further, $\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \left(\frac{b}{a}\right)^n$

(d) $a^m \times a^n = a^{m+n}$ To multiply powers of the same base, **add** the powers.

e.g. $3^2 \times 3^4 = 3^{2+4} = 3^6$ because $3^2 \times 3^4 = (3\times3)\times(3\times3\times3\times3) = 3^6$

(e) $a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$ To divide powers of the same base, **subtract** the powers.

e.g. $3^5 \div 3^2 = 3^{5-2} = 3^3$ because $3^5 \div 3^2 = (3\times3\times3\times3\times3)\div(3\times3) = 3^3$

(f) $\left(a^m\right)^n = a^{m\times n}$ To raise a power to a power, **multiply** the powers.

e.g. $\left(5^2\right)^3 = 5^{2\times3} = 5^6$ because $\left(5^2\right)^3 = 5^2 \times 5^2 \times 5^2 = 5^{2+2+2} = 5^6$

(g) $(a\times b)^n = a^n \times b^n$ To raise a product to a power, raise each number to the power.

e.g. $(3\times5)^2 = 3^2 \times 5^2$ because $(3\times5)^2 = 15^2 = 225$ and $3^2 \times 5^2 = 9\times25 = 225$

(h) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ To raise a quotient to a power, raise each number to the power.

e.g. $\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$ because $\frac{3}{4}\times\frac{3}{4} = \frac{9}{16}$ and $\frac{3^2}{4^2} = \frac{9}{16}$

(i) $a^{\frac{1}{n}} = \sqrt[n]{a}$ A fractional power indicates a root.

e.g. $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ because $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 8^1 = 8$ and $2\times2\times2 = 8$

Further, $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ or $\sqrt[n]{a^m}$

Example 1 Find the value of

(a) 4^{-2} (b) $\left(\frac{2}{3}\right)^{-2}$ (c) $2^3 \times 2^5 \div 2^6$ (d) $27^{\frac{1}{3}}$ (e) $(-8)^{\frac{2}{3}}$

(a) $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$ (b) $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}\left(=2\frac{1}{4}\right)$

(c) $2^3 \times 2^5 \div 2^6 = 2^{3+5-6} = 2^2 = 4$

(d) $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$ or $\left(3^3\right)^{\frac{1}{3}} = 3^{3\times\frac{1}{3}} = 3^1 = 3$

(e) $(-8)^{\frac{2}{3}} = \left(\sqrt[3]{-8}\right)^2 = (-2)^2 = 4$ or $\left\{(-2)^3\right\}^{\frac{2}{3}} = (-2)^{3\times\frac{2}{3}} = (-2)^2 = 4$

(イ) Standard Form

Standard form is a method of expressing numbers in the form $\boxed{a\times10^n}$, where $1 \le a < 10$ and n is an integer. i.e. A number with one digit before the decimal point.

Example 2 Express in standard form: (a) 452 (b) 0.000268

(a) $452 = 4.52 \times 100 = 4.52 \times 10^2$

(b) $0.00268 = 2.68 \times 0.001 = 2.68 \times 10^{-3}$

Example 3 Express as a decimal: (a) 6.45×10^3 (b) 1.07×10^{-2}

(a) $6.45 \times 10^3 = 6.45 \times 1000 = 6450$

(b) $1.07 \times 10^{-2} = 1.07 \times \frac{1}{100} = 0.0107$

(ウ) Approximations

An approximation is a stated value of a number which is close to (but not equal to) the true value of that number.

e.g. 3.14, 3.142 and $\frac{22}{7}$ are all approximations to π (= 3.14159265358979 . . .)

Rounding off: is the process of approximating a number.

> To round off a number, find the place in the number where the rounding off must be done and look at the digit to the right.
> - If this is 5 or greater, add 1 to the rounded off digit.
> - If this is 4 or less, leave the rounded off digit the same.

To a number of decimal places (dp): indicates that rounding off has been done to leave only the number of digits required after the decimal point.

e.g. π = 3.14 (to 2dp) π = 3.142 (to 3dp)

Example 4 Round off 3.0463 correct to: (a) 3dp (b) 2dp (c) 1dp

(a) 3.0463 = 3.046 to 3dp

(b) 3.0463 = 3.05 to 2dp

(c) 3.0463 = 3.0 to 1dp

To a number of significant figures (sf): indicates that rounding off has been done to leave only the number of significant figures required.

e.g. π = 3.14 (to 3sf) π = 3.1416 (to 5sf)

Example 5 Round off 13 507 correct to: (a) 1sf (b) 2sf (c) 3sf (d) 4sf

(a) 13 507 = 10 000 to 1sf

(b) 13 507 = 14 000 to 2sf

(c) 13 507 = 13 500 to 3sf

(d) 13 507 = 13 510 to 4sf

Exercise 3

1 Simplify: (a) $x^{\frac{1}{2}} \times x^{\frac{-1}{3}}$ (b) $x^{-2} \times x^{\frac{1}{2}} \div x^{\frac{-3}{2}}$

2 Find the value of:

(a) $(-5)^{-3}$ (b) $\left(\frac{1}{2}\right)^{-2}$ (c) $5^6 \times 5^{-1} \div 5^3 \times 10^0$ (d) $4^{2.5} \times 4^{-1}$

(e) $\left(-\frac{8}{27}\right)^{\frac{2}{3}}$ (f) $25^{-\frac{1}{2}}$ (g) $\frac{1}{16^{-\frac{3}{4}}}$

simsiky@gmail.com

3 Express in standard form: (a) 703 (b) 8405.2 (c) 0.072 (d) 0.000375
4 Express 2.9735 correct to: (a) 1dp (b) 2dp (c) 3dp
5 Express 40.974 correct to: (a) 1sf (b) 2sf (c) 3sf (d) 4sf

4 Sets

A set is a collection of well defined objects (numbers, letters, symbols and so on).

(ア) Set Notation and Presentation

A set is denoted by a capital letters (such as A).The objects of a set are enclosed in curly brackets, { } and separated by commas.
e.g. V = {a, e, i, o, u}
It is not necessary to list every object in the set. Instead, the rule which the objects follow can be given in the curly brackets.
e.g. V = {vowels in English}

<u>Elements or members</u>: are objects which belong to a set.
The symbol ∈ means "is an element of" or "is a member of". The symbol ∉ means "is not an element of" or "is not a member of".
e.g. 3 ∈ {odd numbers}, 3 ∉ {even numbers}

<u>Universal set</u>: is the set which contains all the elements under discussion.
The universal set is denoted by E.
e.g. If V = {vowels} and C = {consonants} are given, E = {alphabet}

<u>Empty set</u>: is the set which has no elements.
An empty set is denoted by { } or φ.
e.g. {triangles with more than three sides} = { } or φ

<u>Finite and infinite set</u>: A finite set is a set which contains a limited number of elements. An infinite set is a set which contains an unlimited number of elements.
e.g. {letters of the English alphabet} = {a, b, c, . . . , x, y, z} This set is a finite set.
{natural numbers} = {1, 2, 3, 4, . . . } This set is an infinite set.

<u>Number of elements</u>: If a set A is a finite set, the number of elements in the set A is denoted by n(A).
e.g. If A = {letters of the English alphabet}, n(A) = 26

<u>Subset</u>: is a set which also belongs to another set.
The symbol ⊂ means "is a subset of".
e.g. If V = {vowels} and A = {a, e, i}, A ⊂ V.

The number of subsets of a set is 2^n where n is the number of elements in the set.

e.g. A set with 3 elements has $2^3 = 8$ subsets.
∵ If A = {a, b, c}, subsets of A are {a, b, c}, {a, b}, {a, c}, {b, c}, {a}, {b}, {c} and { }

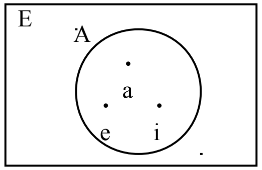
(イ) Operations on Sets

Intersection of sets: is the set of elements common to all the original sets.
The intersection is represented by the symbol $\cap$.
e.g. If A = {1, 2, 3, 4, 5} and B = {2, 4, 6}, $A \cap B$ = {2, 4}

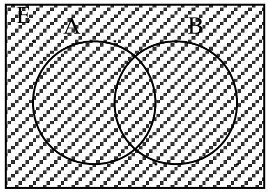
Union of sets: is the set of all the elements of the original sets.
The union is represented by the symbol $\cup$.
e.g. If A = {1, 2, 3, 4, 5} and B = {2, 4, 6}, $A \cup B$ = {1, 2, 3, 4, 5, 6}

Complement of a set: is the set of all elements which are not in that set but are in the universal set originally given.
The complement is represented by the symbol ′.
e.g. If E = {natural numbers less than 10} and A = {prime numbers},
A' = {1, 4, 6, 8, 9} ($\because$ A = {2, 3, 5, 7})

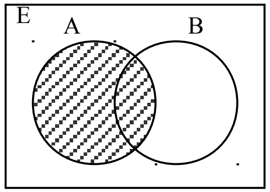
Venn diagrams: are used to show the relationships between sets.
In a Venn diagram, the universal set is represented by a rectangle and sets by circles or simple closed curves. Elements of a set are often represented by points in the circle.



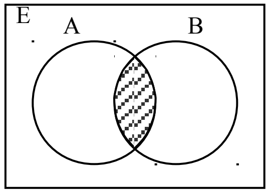
Venn diagrams of some common set relationships
The considered parts are shaded.



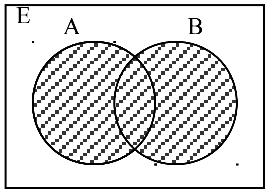
Universal set E



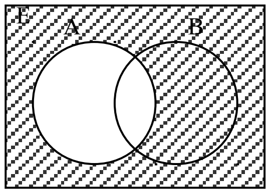
Set A



$A \cap B$
Intersection of A and B



$A \cup B$
Union of A and B



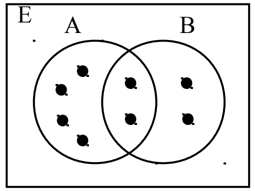
A'
Complement of A

Combined operations

Example 1 For two sets A and B, n(A) = 6, n(B) = 4 and $n(A \cup B)$ = 8. Find $n(A \cap B)$.

n(A) + n(B) = 6 + 4 = 10 which is 2 more than $n(A \cup B)$.

Therefore 2 elements are counted twice, which means that $n(A \cap B) = 2$.

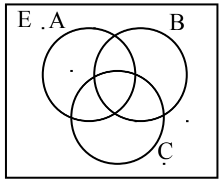


In general, $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

Example 2 If E = {1, 2, 3, 4, 5, 6, 7}, A = {1, 3, 5, 7}, B = {1, 2, 3} and C = {2, 3, 5, 7}, list the elements of: (a) $B' \cap C$ (b) $A \cap C'$ (c) $(A \cup C)'$

(a) B' = {4, 5, 6, 7} so $B' \cap C$ = {5, 7}
(b) C' = {1, 4, 6} so $A \cap C'$ = {1}
(c) $A \cup C$ = {1, 2, 3, 5, 7} so $(A \cup C)'$ = {4, 6}

Example 3 Shade the region $A \cap (B \cup C)'$ on the Venn diagram.



$B \cup C$ is shaded in Fig 1. Then $(B \cup C)'$ is shaded in Fig 2. So $A \cap (B \cup C)'$ is the shaded region in Fig 3.

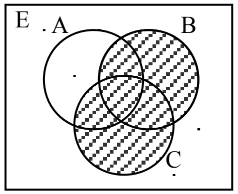


Fig 1

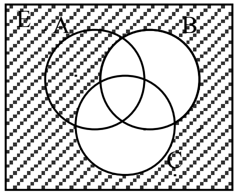


Fig 2

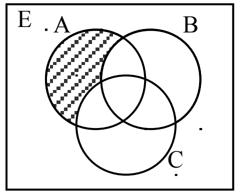
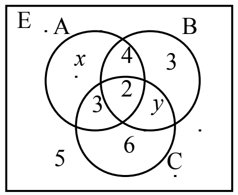


Fig 3

Example 4 If n(E) = 30 and n(A) = 14, find x and y.

$n(A) = x + 4 + 2 + 3 = 14 \quad \therefore x = 5$



simsiky@gmail.com

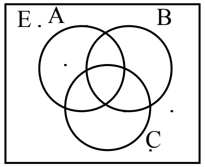
n(E) = 14 + 3 + y + 6 + 5 = 30 (∵n(A) = 14)
∴y = 2

Exercise 4

1 E = {natural numbers less than 10}, A = {prime numbers}, B = {odd numbers} and C = {multiples of 3}. List the elements of (a) A' (b) $A \cap B$ (c) $B' \cap C'$ (d) $A \cup (B \cap C)$

2 If n(A) = 7 and n(B) = 5, what are the largest and smallest possible values of n($A \cup B$) and n($A \cap B$)?

3 A, B and C are three intersecting sets. On separate Venn diagrams, shade the region:
(a) $(A \cup B) \cap C$ (b) $(A \cap C) \cup B$ (c) $A' \cap (B \cup C)$
(d) $A \cap B \cap C'$ (e) $A' \cap B \cap C'$



4 If n(E) = 55, n(A) = 25, n(B) = 22, n(C) = 22, n($A \cap B$) = 8, n($B \cap C$) = 7, n($A \cap C$) = 5 and n($A \cap B \cap C$) = 3, illustrate this information in a Venn diagram.

5 Basic Algebra

1.1 Algebraic Expression

An expression can contain any combination of letters or numbers, and often involves the arithmetic operations.
e.g. $2x$, $6x - 5$, $10 + y$, $x + y$, $3(x - y)$, $3x^2 + 4y$

<u>Variable</u>: is unknown number or quantity represented by a letter (usually such as x, y).

<u>Constant</u>: is a value which is always the same.
e.g. In an expression $y = x + 3$, 3 is a constant.

<u>Coefficient</u>: is a constant which is placed in front of a variable.
e.g. In an expression $x + 2y$, the coefficient of x is 1 and the coefficient of y is 2.
(The absence of a coefficient is equivalent to a 1 being present.)

<u>Terms</u>: are the parts which are separated by a + or – sign.
e.g. An expression $2 + 3x - 4xy + 5x^2$ has 4 terms (2, $3x$, $-4xy$ and.$5x^2$)

<u>Like terms</u>: are terms which contain exactly the same variables and same indices.
They can be collected together by addition or subtraction.
e.g. pairs of like terms are $2x$ and $5x$; $5xy^2$ and $3xy^2$, $2x + 5x = 7x$; $5xy^2 - 3xy^2 = 2xy^2$

<u>Unlike terms</u>: are terms which contain different variables or different indices.
They cannot be added or subtracted.
e.g. pairs of unlike terms are x and x^3; xy^2 and x^2y

<u>Simplification</u>: is combining the terms in an algebraic expression.
<u>Addition and subtraction</u>: Expressions involving addition and subtraction can be simplified by adding or subtracting like terms.
e.g. $3x^2 + 2x + 5y - 2y + 3x + 4x^2 = (3x^2 + 4x^2) + (2x + 3x) + (5y - 2y) = 7x^2 + 5x + 3y$
<u>Multiplication</u>: Expressions involving multiplication can be simplified by multiplying out the terms.
e.g. $4x \times 3y = 4 \times 3 \times x \times y = 12xy$
<u>Division</u>: Expressions involving division can be simplified by cancelling out the terms.

e.g. $$6x^3y^2 \div 2xy = \frac{\overset{3}{\cancel{6}}}{\underset{1}{\cancel{2}}}x^{3-1}y^{2-1} = 3x^2y$$

$$\left(6x^3y^2 \div 2xy = \frac{\overset{3}{\cancel{6}} \times \overset{1}{\cancel{x}} \times x \times x \times \overset{1}{\cancel{y}} \times y}{\underset{1}{\cancel{2}}\,\underset{1}{\cancel{x}}\,\underset{1}{\cancel{y}}} = 3x^2y\right)$$

Substitution: is the replacement of the letters in an expression with specific values.

Example 1 Given that $a = 3$, $b = -5$ and $c = 4$, find the value of: (a) ab^2 (b) $a - b - c^2$

(a) $ab^2 = 3 \times (-5)^2 = 3 \times 25 = 75$

(b) $a - b - c^2 = 3 - (-5) - 4^2 = 3 + 5 - 16 = -8$

Expansion: is removing brackets from an expression by the term immediately before the brackets with every term within them. The distributive law is applied.

e.g. $a(b + c) = ab + ac$

An important type of expansion involves a double set of brackets.

Example 2 Expand $(x + 3)(2x - 5)$

To expand this expression, multiply each term in the first set of brackets with each term in the second set of brackets.

$(x + 3)(2x - 5) = x \times 2x + x \times (-5) + 3 \times 2x + 3 \times (-5)$

$= 2x^2 - 5x + 6x - 15$

$= 2x^2 + x - 15$

1.2 Factorisation

An algebraic expression can be factorised if each term contains one or more common factors. It is written as a product of its factors.

Example 3 Factorise $4x^2 - 12x$

4 and x are the only common factors. Divide the common factor ($4x$) into each term.

$4x^2 \div 4x = x$, $-12x \div 4x = -3$

And write the resulting expression in the brackets.

$4x^2 - 12x = 4x(x - 3)$

Check the answer by expansion.

$4x(x - 3) = 4x \times x + 4x \times (-3) = 4x^2 - 12x$, so the solution is correct.

Example 4 Factorise $3ax + 6ab + 4x + 8b$

There is no common factor to all the four terms. Take them in pairs.

$3ax + 6ab + 4x + 8b = 3a(x + 2b) + 4(x + 2b)$

($\because 3a$ is the common factor to the first two terms and 4 is the common factor to the last two terms.)

Now $(x + 2b)$ is the common factor. $\therefore$expression $= (x + 2b)(3a + 4)$

Factorisation of quadratic expressions: A quadratic expression ($ax^2 + bx + c$) can be factorised into two pairs of brackets by using the following method.

Step 1: find the product of a and c.
Step 2: find the factors of the product of a and c, whose sum is b.
Step 3: express the term bx as a sum of terms, using the factors in step 2.
Step 4: factorise by grouping terms with common factors.

Example 5 Factorise $x^2 + 5x + 6$

p	6
s	5
f	2, 3

The product is $1 \times 6 = 6$ and the sum is 5.

6 could be 1×6 or 2×3.

$2 + 3 = 5$ ($1 + 6 = 7 \neq 5$) so the factors are 2 and 3.

$x^2 + 5x + 6 = x^2 + 2x + 3x + 6 = x(x + 2) + 3(x + 2) = (x + 2)(x + 3)$

Only if the coefficient of x^2 is 1, the expression can be factorised **directly** by using the factors.

$$\boxed{x^2 + (a + b)x + ab = (x + a)(x + b)}$$

Example 6 Factorise $3x^2 - x - 10$

p	–30
s	–1
f	–6, 5

The product is $3 \times (-10) = -30$ and the sum is -1.

30 could be 1×30, 2×15, 3×10 or 5×6.

Now the product is negative, so one factor is positive and the other is negative.

$-6 + 5 = -1$ so the factors are -6 and 5

$3x^2 - x - 10 = 3x^2 - 6x + 5x - 10 = 3x(x - 2) + 5(x - 2) = (x - 2)(3x + 5)$

Always check the answer by expansion.

$(x - 2)(3x + 5) = x \times 3x + x \times 5 - 2 \times 3x - 2 \times 5 = 3x^2 + 5x - 6x - 10 = 3x^2 - x - 10$

simsiky@gmail.com

Difference of two squares: is an expression of the form $a^2 - b^2$ which can be factorised to $(a + b)(a - b)$. ~~$a^2 - b^2 = (a + b)(a - b)$~~

Example 7 Factorise $4x^2 - 9$

$4x^2 - 9 = (2x)^2 - 3^2 = (2x + 3)(2x - 3)$

1.3 Algebraic Fractions

A fraction which contains at least one letter is called an algebraic fraction.

Algebraic fractions can be simplified to their lowest terms by factorising both the denominator and numerator.

Example 8 Simplify: (a) $\frac{x^2}{4x + xy}$ (b) $\frac{x+1}{x^2+3x+2}$

(a) $\frac{x^2}{4x+xy} = \frac{x^2}{x(4+y)} = \frac{x}{4+y}$ (b) $\frac{x+1}{x^2+3x+2} = \frac{x+1}{(x+1)(x+2)} = \frac{1}{x+2}$

Addition and subtraction: Algebraic fractions can be added or subtracted in exactly the same way as arithmetical fractions by expressing them as fractions with the lowest common multiple (LCM) as the common denominator.

Example 9 Express as a single fraction and simplify where possible.

(a) $\frac{x+1}{3} + \frac{x+3}{2}$ (b) $\frac{3}{x-2} - \frac{2}{x-3}$ (c) $\frac{1}{x^2+3x+2} + \frac{2x}{x^2-4} + \frac{1}{x^2-x-2}$

(a) The LCM of 3 and 2 is 6.

$$\frac{x+1}{3} + \frac{x+3}{2} = \frac{2(x+1)}{6} + \frac{3(x+3)}{6} = \frac{2x+2+3x+9}{6} = \frac{5x+11}{6}$$

(b) The LCM of $(x - 2)$ and $(x - 3)$ is $(x - 2)(x - 3)$.

$$\frac{3}{x-2} - \frac{2}{x-3} = \frac{3(x-3)}{(x-2)(x-3)} - \frac{2(x-2)}{(x-2)(x-3)} = \frac{3(x-3)-2(x-2)}{(x-2)(x-3)}$$

$$= \frac{3x-9-2x+4}{(x-2)(x-3)} = \frac{x-5}{(x-2)(x-3)}$$

(c) First factorise the denominators each and then find the LCM.

$x^2 + 3x + 2 = (x + 1)(x + 2)$

$x^2 - 4 = (x + 2)(x - 2)$

$x^2 - x - 2 = (x + 1)(x - 2)$

$\therefore$ The LCM $= (x + 1)(x + 2)(x - 2)$

$$\frac{1}{(x+1)(x+2)} + \frac{2x}{(x+2)(x-2)} + \frac{1}{(x+1)(x-2)}$$

$$= \frac{1\times(x-2)}{(x+1)(x+2)(x-2)} + \frac{2x\times(x+1)}{(x+1)(x+2)(x-2)} + \frac{1\times(x+2)}{(x+1)(x+2)(x-2)}$$

$$= \frac{x-2+2x^2+2x+x+2}{(x+1)(x+2)(x-2)} = \frac{2x^2+4x}{(x+1)(x+2)(x-2)} = \frac{2x(x+2)}{(x+1)(x+2)(x-2)}$$

$$= \frac{2x}{(x+1)(x-2)}$$

Multiplication and division: Algebraic fractions can be multiplied or divided in exactly the same way as arithmetical fractions.

Example 10 Simplify where possible.

(a) $\frac{2z}{xy} \times \frac{y}{4z}$ (b) $\frac{4}{a+b} \times \frac{a^2-b^2}{12}$ (c) $\frac{xy}{6} \div \frac{y}{3x}$ (d) $\frac{x^2-4}{3x^2-6x} \div \frac{x^2+x-2}{9x^2}$

(a) $\frac{2z}{xy} \times \frac{y}{4z} = \frac{2z \times y}{xy \times 4z} = \frac{1}{2x}$

(b) First $a^2 - b^2$ is factorised to $(a + b)(a - b)$

$$\frac{4}{a+b}\times\frac{(a+b)(a-b)}{12}=\frac{4\times(a+b)(a-b)}{(a+b)\times 12}=\frac{a-b}{3}$$

(c) First convert division to multiplication.

$$\frac{xy}{6}\div\frac{y}{3x}=\frac{xy}{6}\times\frac{3x}{y}=\frac{xy\times 3x}{6\times y}=\frac{x^2}{2}$$

(d) Convert division to multiplication and factorise.

$$\frac{x^2-4}{3x^2-6x}\div\frac{x^2+x-2}{9x^2}=\frac{(x+2)(x-2)}{3x(x-2)}\times\frac{9x^2}{(x+2)(x-1)}$$

$$=\frac{(x+2)\times 9x^2}{3x\times(x+2)(x-1)}=\frac{3x}{x-1}$$

Exercise 5

1 Expand and simplify where possible: (a) $2x(3x-y)$ (b) $3(3x-y)-4(4y+x)$

(c) $(a+5)(a-2)$ (d) $(x-2y)(3x+4y)$ (e) $(2a-3b)^2$ (f) $\left(2x-\frac{1}{x}\right)^2$

2 Factorise completely: (a) $3x^2-6x$ (b) $3ac-6ad-2bc+4bd$ (c) x^2+3x+2

(d) x^2+6x+9 (e) $x^2-4x-21$ (f) $2x^2-5x-3$ (g) $12x^2-5xy-2y^2$

(h) $49-16x^2$ (i) $a^2b^2-4c^2$ (j) $st^2-4s+2t^2-8$

3 Simplify: (a) $\frac{xyz}{xyz^2}$ (b) $\frac{3x^2+10x-8}{6x^2-13x+6}$ (c) $\frac{2-a}{a^2-4}$ (d) $\frac{1}{2a}+\frac{2}{a}$

(e) $\frac{1}{x+y}+\frac{1}{x-y}$ (f) $\frac{1}{x-2}-\frac{1}{2x^2-7x+6}$ (g) $\frac{2x-2y}{9}\times\frac{3}{y-x}$

(h) $\frac{x+1}{x^2-7x+12}\times\frac{2x-8}{x^2+2x+1}$ (i) $\frac{x^2+4xy+3y^2}{xy}\div\frac{x+y}{y}$

6 Equations

An algebraic equation is a mathematical statement that two algebraic expressions are equal. An equation is solved by finding the value of the unknown variable(s). Any value of the variable(s) that satisfies the equation is a solution.

1.4 Rearranging an Equation (Changing the subject of an equation)

Expressions in an equation can be rearranged so that one of the terms is on its own to the left of the equal sign (=). Then the equation can be solved for that term.

simsiky@gmail.com

Example 1 Rearrange for x. (a) $x + 4y = 3$ (b) $y = \frac{2x-1}{x-3z}$

(a) Leave x on the left side and move $4y$ to the right side. $\therefore x = 3 - 4y$

(b) Remove the denominator $(x - 3z)$ by multiplying both sides by it.

$y(x - 3z) = 2x - 1$ i.e. $xy - 3yz = 2x - 1$

Collect the x terms by themselves on the left side.

$xy - 2x = -1 + 3yz$ i.e. $x(y - 2) = 3yz - 1$

Divide both sides by $(y - 2)$. $\therefore x = \frac{3yz-1}{y-2}$

(ア) Linear Equations

A linear equation is an equation involving an expression of the first degree (the highest power of variables is 1).

Linear equations in one variable: A linear equation in one variable x can be written in the form $ax + b = 0$, where $a \neq 0$. e.g. $x + 2 = 7$, $3x + 2 = 5$

To solve a linear equation in one variable, rearrange and solve it for that variable.

Example 2 Solve for x: (a) $2x - 1 = x + 3$ (b) $7(x + 3) - 2(x - 4) = 4$

(c) $\frac{x-5}{2} - \frac{3x+1}{5} = -3$ (d) $\frac{3}{2x} - \frac{1}{x} = \frac{1}{4}$

(a) Collect like terms. $2x - x = 3 + 1$ $\therefore x = 4$

Check the answer by substituting it into each side of the original equation.

Left side = $2 \times 4 - 1 = 8 - 1 = 7$ Right side = $4 + 3 = 7$

As left side = right side, the answer is correct.

(b) First expand. $7x + 21 - 2x + 8 = 4$

Then collect like terms and simplify. $5x = 4 - 29$ $5x = -25$

Divide both sides by 5. $x = -5$

(c) First remove fractions by multiplying the LCM (10).

$10 \times \frac{x-5}{2} - 10 \times \frac{3x+1}{5} = 10 \times (-3)$ $5(x - 5) - 2(3x + 1) = -30$

Then expand, collect like terms and simplify.

$5x - 25 - 6x - 2 = -30$ $-x = -30 + 27$ $-x = -3$ $\therefore x = 3$

(d) Remove fractions by multiplying the LCM ($4x$).

$4x \times \frac{3}{2x} - 4x \times \frac{1}{x} = 4x \times \frac{1}{4}$ $6 - 4 = x$ $\therefore x = 2$

Linear equations in two variables: A linear equation in two variables x and y can be written in the form $ax + by + c = 0$, where a and b are not equal to 0.

e.g. $x - y = 3$, $2x + y = 5$, $2x - 4y - 9 = 0$

These equation have an infinite number of pairs of values of x and y which satisfy the equations.

Simultaneous equations of linear equations in two variables: are pairs of linear equations in which the two variables represent the same numbers in each equation.

To solve simultaneous equations, find a solution which satisfies both equations.

There are mainly two methods to solve simultaneous equations.

(i) Substitution method

(ii) Elimination method

Unless the question asks for a specific method, you are free to use any method you prefer.

Substitution method: One variable is expressed in terms of the other in either of the given equations and this expression is substituted into the other equation.

Example 3 Solve: $2x - y = 5$

$x - 2y = 4$

Rearrange the second equation to solve for x. $x = 4 + 2y$

Substitute this expression for x into the first equation. $2(4 + 2y) - y = 5$

Expand, collect like terms and simplify.

$8 + 4y - y = 5$ $3y = 5 - 8 = -3$ $\therefore y = -1$

Substitute the value of y into the rearranged equation. $x = 4 + 2 \times (-1) = 4 - 2 = 2$

The solution is $x = 2$, $y = -1$

Check the solution by substituting it into the other equation.

Left side = $2x - y = 2 \times (2) - (-1) = 4 + 1 = 5$ = Right side, so the solution is correct.

Elimination method: One variable is eliminated in order to leave only the other variable.

Example 4 Solve: (a) $2x - 3y = 4$ (b) $x + 4y = 7$ (c) $2x - y = 5$

$x + 3y = 11$ $x + 2y = 3$ $x - 4y = 6$

(a) Now the coefficients of y are opposite (– 3 and 3), add the equations.

$2x - 3y = 4$

$+$

$x + 3y = 11$

$(2x + x) + (-3y + 3y) = 4 + 11 \quad 3x = 15 \quad \therefore x = 5$

Substitute 5 for x into either of the equation (now the second equation).

$5 + 3y = 11 \quad 3y = 11 - 5 = 6 \quad \therefore y = 2$

The solution is $x = 5$, $y = 2$.

Always Check the solution by substituting it into the other equation.

Left side = $2 \times 5 - 3 \times 2 = 10 - 6 = 4$ = Right side, so the solution is correct.

(b) Now the coefficients of x are equal (1 and 1), subtract one equation from the other.

$x + 4y = 7$

$-\ x + 2y = 3$

$(x - x) + (4y - 2y) = 7 - 3 \quad 2y = 4 \quad \therefore y = 2$

Substitute 2 for y into either of the equation (now the second equation).

$x + 2 \times 2 = 3 \quad x = 3 - 4 = -1$

The solution is $x = -1$, $y = 2$

(c) If the coefficients are not equal or opposite, find the LCM of the coefficients of one of the variables and make them the same in each equation.

Now choose x, the LCM of 1 and 2 is 2. Multiply the second equation by 2.

$2 \times (x - 4y = 6) \quad 2x - 8y = 12$

The coefficients of x are equal (2 and 2), subtract one equation from the other.

$2x - y = 5$

$-\ 2x - 8y = 12$

$(2x - 2x) + (-y - (-8y)) = 5 - 12 \quad 7y = -7 \quad \therefore y = -1$

Substitute -1 for y into either of the equation (now the second equation).

$x - 4 \times (-1) = 6 \quad x = 6 - 4 = 2$

The solution is $x = 2$, $y = -1$

(イ) Quadratic Equations

A quadratic equation in one variable x can be written in the form $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$. e.g. $x^2 + 3x + 2 = 0$, $x^2 - 4 = 0$

The solutions of a quadratic equation are called roots. Quadratic equations will always have **two** roots which may be equal.

There are three methods to solve quadratic equations.

(i) Factorisation method

(ii) Completing the square method

(iii) Formula method

Factorisation method: involves factorising the equation to give two expressions in brackets. In general, given $ab = 0$, we can say $a = 0$ or $b = 0$. So by taking each bracket at a time, the two possible solutions can be found. Not all quadratic equations can be solved by factorisation method.

Example 5 Solve: (a) $x^2 - 5x = 0$ (b) $x^2 - 3x + 2 = 0$

(a) Factorise the left side. $x(x - 5) = 0$

Since the product of the factors is 0, one of the factors must be 0.

$x = 0$ or $x - 5 = 0$

Therefore the roots are $x = 0$ or $x = 5$.

Check the solutions by substituting them into the original equation.

When $x = 0$, Left side = $0^2 - 5 \times 0 = 0 - 0 = 0$ = Right side

When $x = 5$, Left side = $5^2 - 5 \times 5 = 25 - 25 = 0$ = Right side

So the solutions are correct.

(b) Factorise the left side. $(x - 1)(x - 2) = 0$

(In the quadratic equation, the product is 2 and the sum is -3, so factors are -1 and -2.)

$x - 1 = 0$ or $x - 2 = 0$

Therefore the roots are $x = 1$ or $x = 2$.

This method should always be tried first. If it is not possible, then use either of the other two methods.

Completing the square method: is making the left side of a quadratic equation into a perfect square, the form $(x + a)^2$. This method can be used to solve any quadratic equation.

Example 6 Solve $x^2 - 4x - 1 = 0$

$x^2 - 4x - 1 = 0$ cannot be factorised.

simsiky@gmail.com

Move the constant (–1) to the right side. $x^2 - 4x = 1$ (i)

Halve the coefficient of x and complete the square on the left side.

$$\left(x - \frac{4}{2}\right)^2 = (x-2)^2 = x^2 - 4x + 4 \quad \therefore x^2 - 4x = (x-2)^2 - 4$$

So (i) becomes $(x-2)^2 - 4 = 1$ which gives $(x-2)^2 = 5$

Take the square root of both sides. $x - 2 = \pm\sqrt{5} = \pm 2.236$

So $x = 2 + 2.236$ or $x = 2 - 2.236$

Therefore $x = 4.24$ (2dp) or $x = -0.24$ (2dp)

Formula method: is given by generalising the above method.

Take the general quadratic equation as $ax^2 + bx + c = 0$.

Divide both sides by a. $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Move the constant ($\frac{c}{a}$) to the right side. $x^2 + \frac{b}{a}x = -\frac{c}{a}$ (ii)

Now $\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$

So (ii) becomes: $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = -\frac{c}{a}$ $\quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ Then $\boxed{x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}}$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 7 Solve $2x^2 - 5x + 1 = 0$

Compare with $ax^2 + bx + c = 0$: $a = 2$, $b = -5$ and $c = 1$

Then $x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 2 \times 1}}{2 \times 2} = \frac{5 \pm \sqrt{17}}{4} = \frac{5 \pm 4.123}{4}$

$= \frac{9.123}{4}$ or $\frac{0.877}{4} = 2.28$ or 0.22 (2dp)

(ウ) Equations Involving Indices

There are two types of equations involving indices.

Equations of the form $x^a = b$: (where x is a variable and a and b are constants) can be solved by raising both sides of the equation to the reciprocal of the power a.

In general,, $x^a = b$

The reciprocal of a is $\frac{1}{a}$. $\left(x^a\right)^{\frac{1}{a}} = b^{\frac{1}{a}}$ $\quad x^{a \times \frac{1}{a}} = b^{\frac{1}{a}}$ $\quad \therefore x = b^{\frac{1}{a}}$

Example 8 Solve: (a) $x^3 = 8$ (b) $x^{\frac{1}{2}} = 3$ (c) $x^{\frac{-2}{3}} = 16$

(a) The reciprocal of 3 is $\frac{1}{3}$. $\left(x^3\right)^{\frac{1}{3}} = 8^{\frac{1}{3}}$ $\quad x^{3 \times \frac{1}{3}} = \left(2^3\right)^{\frac{1}{3}}$

$\therefore x = 2^{3 \times \frac{1}{3}} = 2^1 = 2$ (or $x = 8^{\frac{1}{3}} = \sqrt[3]{8} = 2$)

(b) The reciprocal of $\frac{1}{2}$ is 2. $\left(x^{\frac{1}{2}}\right)^2 = 3^2$ $\quad x^{\frac{1}{2} \times 2} = 9$ $\quad \therefore x = 9$

(c) The reciprocal of $-\frac{2}{3}$ is $-\frac{3}{2}$. $\left(x^{\frac{-2}{3}}\right)^{\frac{-3}{2}} = 16^{\frac{-3}{2}}$ $\quad x^{\frac{-2}{3} \times \left(\frac{-3}{2}\right)} = \left(4^2\right)^{\frac{-3}{2}}$

$\therefore x = 4^{2 \times \left(\frac{-3}{2}\right)} = 4^{-3} = \frac{1}{4^3} = \frac{1}{64}$ (or

$x = 16^{\frac{-3}{2}} = \left(16^{\frac{1}{2}}\right)^{-3} = \left(\sqrt{16}\right)^{-3} = 4^{-3}$)

Equations of the form $a^x = b$: (where x is a variable and a and b are constants) can be solved by expressing both sides of the equation in index form with **same base**, and equating the powers.

Example 9 Solve: (a) $2^x = 8$ (b) $25^x = 5$ (c) $27^x = \frac{1}{9}$

(a) Express both sides in index form with common base. $2^x = 2^3$

Then equate powers. $x = 3$

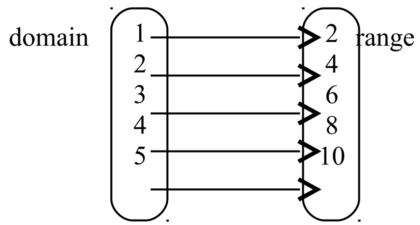
(b) $\left(5^2\right)^x = 5^1 \quad 5^{2x} = 5^1 \quad 2x = 1 \quad \therefore x = \frac{1}{2}$

(c) $\left(3^3\right)^x = \frac{1}{3^2} \quad 3^{3x} = 3^{-2} \quad 3x = -2 \quad \therefore x = -\frac{2}{3}$

(工) Functions

A function is an operation that is applied to a given set (the domain) of values to give another set (the range) of values. Each element in the domain is related to only one element in the range. A function is represented by the letter *f*.

e.g. $f(x) = 2x$



Example 10 If $f(x) = 2x + 3$, find: (a) $f(-2)$ (b) x when $f(x) = 5$

(a) Substitute –2 for x into the equation.

$f(-2) = 2 \times (-2) + 3 = -4 + 3 = -1$

(b) Substitute 5 for $f(x)$ into the equation and then solve for x.

$5 = 2x + 3 \quad 5 - 3 = 2x \quad 2 = 2x \quad \therefore x = 1$

Notation

$y = f(x)$ is a way of expressing that there is a function of x which maps x-numbers onto y-numbers.

e.g. Given that $f(x) = 2x + 3$, then $y = f(x)$ is the same as $y = 2x + 3$

$f(x) = y$ is sometimes written as $f: x \to y$.

e.g. $f: x \to 2x$ is the same as $f(x) = 2x$

Inverse functions: An inverse function is an operation that reverses a function. This is written as $f^{-1}(x)$. i.e. If f maps x onto y, then f^{-1} maps y onto x.

The inverse of a function can be found by rearranging an equation for x.

Example 11 Find: (a) the inverse function of $f(x) = \frac{3-5x}{2}$ (b) $f^{-1}(-1)$

(a) Let $y = f(x)$. $y = \frac{3-5x}{2}$

Rearrange for x. $2y = 3 - 5x \quad 5x = 3 - 2y \quad \therefore x = \frac{3-2y}{5}$

Exchange x and y. $y = \frac{3-2x}{5}$

Let $y = f^{-1}(x)$. $\therefore f^{-1}(x) = \frac{3-2x}{5}$

(b) Substitute –1 for x into the equation. $f^{-1}(-1) = \frac{3-2\times(-1)}{5} = \frac{5}{5} = 1$

Exercise 6

1 Rearrange the following equations to make the subject the letter in brackets.

(a) $I = \frac{PRT}{100}$ [R] (b) $V = \frac{3}{4}\pi r^3$ [r] (c) $p = \frac{q-t}{2t}$ [t] (d) $y = \sqrt{\frac{x+1}{3x-2}}$ [x]

2 Solve for x. (a) $10x + 3 = 4x + 6$ (b) $3(x - 1) - (1 - x) = -4$

(c) $\frac{4(x-2)}{5} - \frac{x+3}{2} = -\frac{1}{10}$ (d) $\frac{2}{3x} - \frac{1}{2x} = \frac{1}{4}$

3 Solve the simultaneous equations.

(a) $x + y = 9$, $y = 2x$ (b) $2y - 3x = -22$, $x = 4 - y$ (c) $x + 2y = 0$, $x - 3y = 2$ (d) $2x + 3y = 4$, $3x - 2y = -7$

4 Solve for x. Where necessary give the answers correct to 2 decimal places.

(a) $x^2 = 3x$ (b) $x^2 + 5x + 6 = 0$ (c) $x^2 - 6x = -9$ (d) $2x^2 - 5x - 3 = 0$

(e) $3x^2 - x - 1 = 0$ (f) $2x^2 + x = 5$

5 Solve for x: (a) $x^3 = -27$ (b) $x^{\frac{2}{3}} = 4$ (c) $x^4 = \frac{1}{16}$ (d) $x^{-3} = 8$

simsiky@gmail.com

(e) $2^x = 16$ (f) $10^x = 1$ (g) $25^x = \frac{1}{5}$ (h) $\left(\frac{1}{16}\right)^x = 8$

6 Given that $f(x) = \frac{2x-3}{x}$, $x \neq 0$,

find: (a) $f(-3)$ (b) x when $f(x) = 5$ (c) an expression of $f^{-1}(x)$ (d) $f^{-1}(-4)$

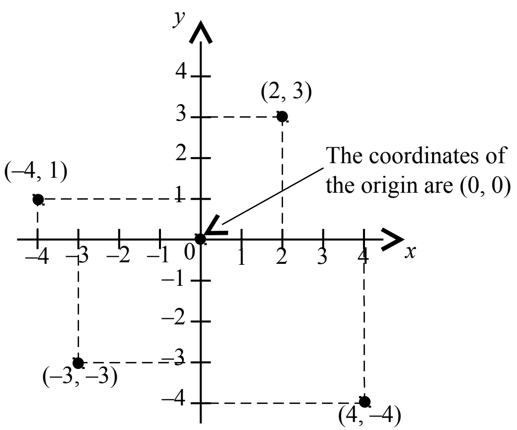
7 Coordinates and Graphs

(ア) Coordinate Systems

Cartesian coordinates: give the position of a point in a plane (two dimensions) by reference to two coordinates axes (the x-axis and the y-axis) at right angles

The coordinates (x, y) describe the position of a point in terms of the distance of the point from the origin, (0, 0). i.e. The x-coordinate is the distance of the point from the origin, parallel to the x-axis and the y-coordinate is the distance of the point from the origin, parallel to the y-axis. The x-coordinate is always written first.

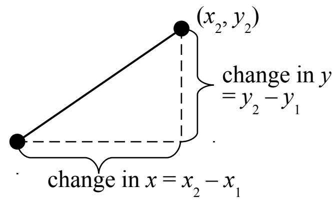
e.g.



(イ) General Graph Terms

Gradient: is the rate at which y increases compared with x between any two points. The gradient is usually written as m.

$$\text{gradient } (m) = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

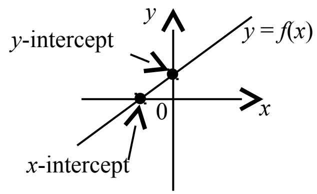


A positive gradient slopes upward to the right.

A negative gradient slopes downward to the right.

***x*-intercept:** is the point where the line or curve cuts across the x-axis. At the x-intercept, $y = 0$.

***y*-intercept:** is the point where the line or curve cuts across the y-axis. At the y-intercept, $x = 0$.



(ウ) Straight Line Graphs

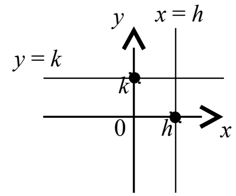
Constant graphs: are graphs in which the value of a variable is constant.

e.g. $x = h \rightarrow$ The line is parallel to y-axis.

At any point on the line, x-coordinate is h.

$y = k \rightarrow$ The line is parallel to x-axis.

At any point on the line, y-coordinate is k.

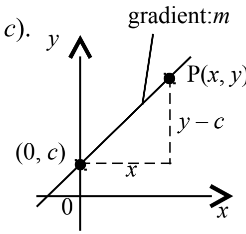


Linear graphs: are graphs in which the relationship between variables is given by a linear equation.

In the diagram, the line has gradient m and cuts the y-axis at $(0, c)$. c is the y-intercept. $P(x, y)$ is any point on the line.

Then $m = \frac{y-c}{x}$ $\therefore$ $\boxed{y = mx + c}$

This is the general equation of a straight line which has a



gradient m and the y-intercept c.

Example 1 State the gradient and y-intercept of the line $x + 2y = 2$ and draw the line.

Rearrange for y. $2y = -x + 2$ $\therefore\ y = -\dfrac{1}{2}x + 1$

Then gradient $= -\dfrac{1}{2}$ and y-intercept = 1.

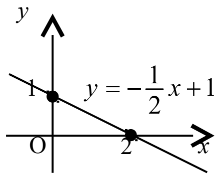
To draw a straight line, the coordinates of any **two points** on the line must be found.

(In most cases, x-intercept and y-intercept are useful.)

Find the x-intercept. At the x-intercept, $y = 0$. $x + 2 \times 0 = 2$ $\therefore x = 2$

The coordinates of the two points are (0, 1) and (2, 0).

Then join them.



Example 2 Find the equation of the line through A(1, 7) and B(–2, –2).

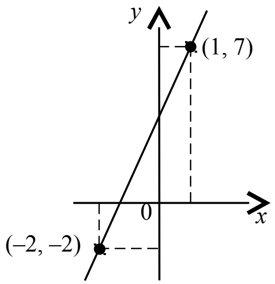
The gradient $m = \dfrac{7-(-2)}{1-(-2)} = \dfrac{9}{3} = 3$

The equation is $y = 3x + c$

Then find the y-intercept c by substituting $x = 1$ and $y = 7$ into the equation.

$7 = 3 \times 1 + c$ $c = 7 - 3 = 4$

So the equation is $y = 3x + 4$



(エ) Curve Graphs

Quadratic graphs: are graphs in which the relationship between variables is given a quadratic equation. All quadratic graphs can be written in the form $y = ax^2 + bx + c$. Its shape is that of a parabola.

Example 3 The table below shows some of the values of x and the corresponding values of y for the equation $y = -2x^2 - x + 8$.

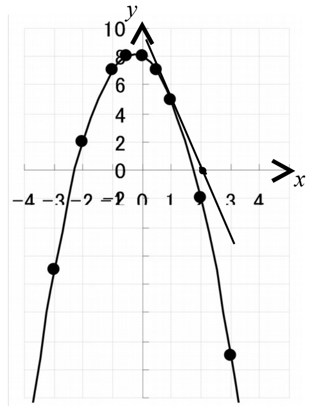
x	–3	–2	–1	–1/2	0	1/2	1	2	3
y	–7	2	7	8	8	7	5	–2	–13

(a) Draw the graph.

(b) By drawing a tangent, find the gradient of the curve at the point (1, 5).

(A tangent is the line which touches the curve at the point.)

(c) Estimate the area bounded by the curve, the x-axis, $x = -1$ and $x = 1$.

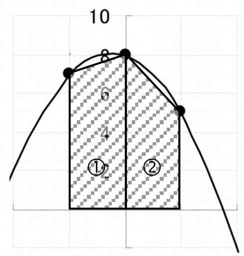


(a) Plot these coordinates and draw a smooth curve through them.

(b) Draw a tangent and find the coordinates of any point on the line. Now use the x-intercept (2, 0).

So the gradient $m = \dfrac{0-5}{2-1} = -5$

(c) Draw two trapeziums in which each height is 1 unit. Then find the total area of the trapeziums.



Area = area of ① + area of ②

$= \dfrac{1}{2} \times (7+8) \times 1 + \dfrac{1}{2} \times (5+8) \times 1 = 7.5 + 6.5 = 14\ \text{unit}^2$

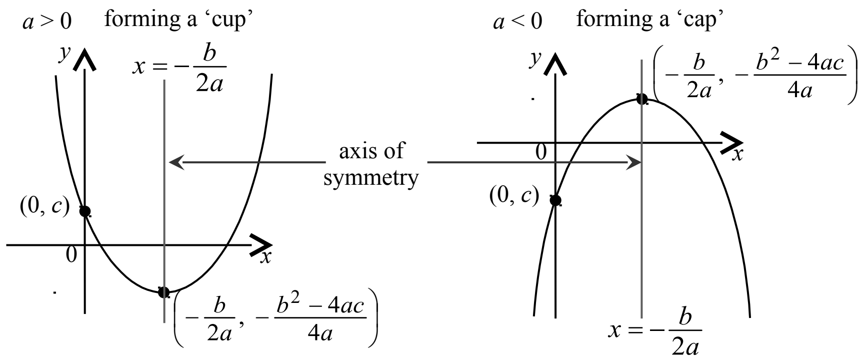
If each height of a trapezium is 1/2 unit, a better estimate is obtained.

In general, completing the square

$$y = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

This gives the coordinates of the turning point $\left(-\dfrac{b}{2a},\ -\dfrac{b^2 - 4ac}{4a}\right)$.

simsiky@gmail.com



In general, to draw a quadratic graph ($y = ax^2 + bx + c$), find the coordinates of:

(i) the y-intercept = (0, c)

(ii) the x-intercepts = (x, 0) (if it exists and if the expression can be factorised)

(the values of x are the roots of the equation $ax^2 + bx + c = 0$)

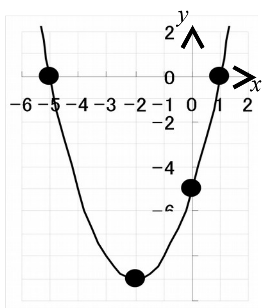
(iii) the turning point (the bottom of a parabola) = $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$

Example 4 Draw the graph of the equation $y = x^2 + 4x - 5$.

First find the y-intercept. $c = -5$

Find the x-intercept. At the x-intercept, $y = 0$

$0 = x^2 + 4x - 5$ $(x + 5)(x - 1) = 0$ $\therefore x = -5, 1$



Find the turning point by completing the square.

$y = (x + 2)^2 - 4 - 5 = (x + 2)^2 - 9$

So the coordinates of the turning point are (–2, –9)

Plot these points and draw a smooth curve through them.

Other graphs

Example 5 The variables x and y are connected by the equation $y = 20 - \frac{12}{x} - x^2$.

The table below shows some corresponding values of x and y. The values of y are given correct to one decimal place where appropriate.

x	1	1.5	2	3	4	5	6	7
y	7	9.8	10	7	1	–7.4	–18	a

(a) Calculate the value of a.

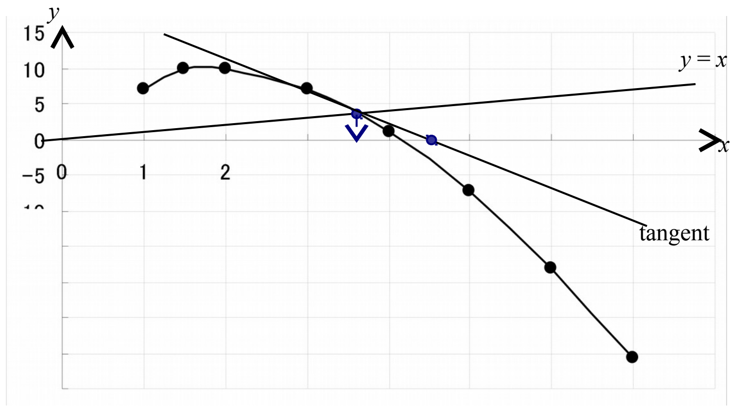
(b) Draw the graph of $y = 20 - \frac{12}{x} - x^2$ for the range of $1 \le x \le 7$.

(c) By drawing a tangent, find the gradient of the curve at the point where $x = 3$.

(d) Using your graph, solve the equation $20 - \frac{12}{x} - x^2 = x$

(a) Substitute 7 for x into the equation. $a = 20 - \frac{12}{7} - 7^2 = -30.7$ (to 1dp)

(b) Plot the points and draw a smooth curve.



(c) Draw a tangent and then find the coordinates of the x-intercept. $x \approx 4.5$

$\therefore$ the gradient $m = \dfrac{0-7}{4.5-3} = -\dfrac{14}{3}$

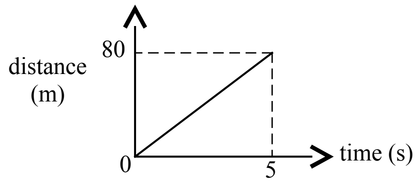
(d) The solution is obtained by solving the simultaneous equations $y = 20 - \dfrac{12}{x} - x^2$ and $y = x$. Draw the line $y = x$. The point where two lines intersect is the solution. $\therefore x \approx 3.6$

(才) Distance-Time and Speed-Time Graphs

Distance-time graphs: are graphs which show the distance moved by an object with time.

> The **gradient** gives the rate of change in distance i.e. the **speed**.
>
> $$\text{speed} = \frac{\text{change in distance}}{\text{time}}$$

Example 6 The graph below shows the distance of an object over a time 5sec. Find the speed.



$$\text{speed} = \frac{\text{change in distance}}{\text{time}} = \frac{80-0}{5} = 16\text{ m/s}$$

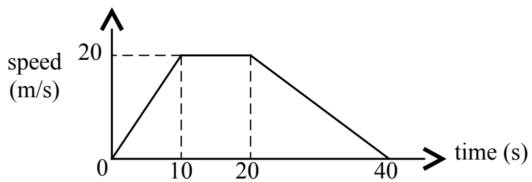
Speed-time graphs: are graphs which show the speed of an object with time.

> The **gradient** gives the rate of change in speed i.e. the **acceleration**.
>
> $$\text{acceleration} = \frac{\text{change in speed}}{\text{time}}$$

If acceleration is negative, it is called deceleration or retardation. This means the speed of an object decreases.

> The **area** under the speed-time graph gives the **distance** travelled by an object.

Example 7 The diagram below is a speed-time graph of a journey.



Calculate: (a) the acceleration for the first 10sec.

(b) the deceleration for the last 20sec.

(c) the distance covered in the first 10sec.

(d) the total distance covered in 40sec.

(e) the average speed for the whole journey.

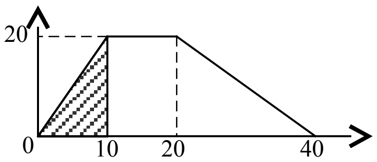
(f) the speed at 25sec.

(a) $\text{acceleration} = \dfrac{\text{change in speed}}{\text{time}} = \dfrac{20-0}{10} = 2\text{ m/s}^2$

(b) The formula of deceleration is the same as that of acceleration.

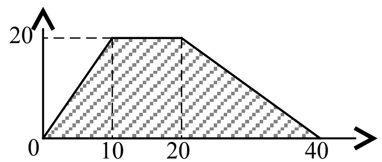
$$\text{deceleration} = \frac{\text{change in speed}}{\text{time}} = \frac{0-20}{20} = -1\text{ m/s}^2$$

(c) The distance covered in the first 10sec is represented by the shaded area below.



$$\text{distance} = \text{shaded area} = \frac{1}{2} \times 10 \times 20 = 100\text{ m}$$

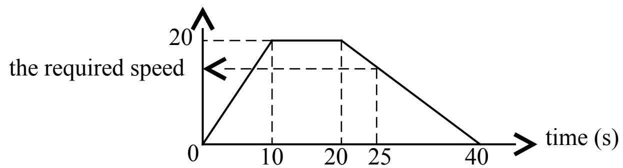
(d) The total distance covered is represented by the shaded area below.



$$\text{distance} = \text{shaded area} = \frac{1}{2} \times (10+40) \times 20 = 500\text{ m}$$

(e) $\text{average speed} = \dfrac{\text{total distance}}{\text{total time}} = \dfrac{500}{40} = 12.5\text{ m/s}$

(f) This question asks the speed as shown below.



simsiky@gmail.com

From the formula of acceleration $\left(a = \dfrac{v-u}{t}\right)$, $v = u + at$ is obtained.

Now $a = -1$ m/s^2 from (b).

∴ speed = 20 + (–1) × 5 = 20 – 5 = 15 m/s

Exercise 7

1 Draw the following lines and write down: (i) the gradient, (ii) the y-intercept each.
(a) $y = x - 2$ (b) $2x - y = -4$ (c) $2x + 3y = 6$

2 Draw the following quadratic graph: (a) $y = 2x^2$ (b) $y = -x^2 + 3x + 10$

3 The variables x and y are connected by the equation $y = 4 - \dfrac{6}{x} + x$ and some corresponding values are given in the table below.

x	0.5	1	1.5	2	3	4	5
y	–7.5	–1	1.5	3	5	6.5	a

(a) Calculate the value of a.

(b) Draw the graph of $y = 4 - \dfrac{6}{x} + x$ for the range of $0.5 \le x \le 5$.

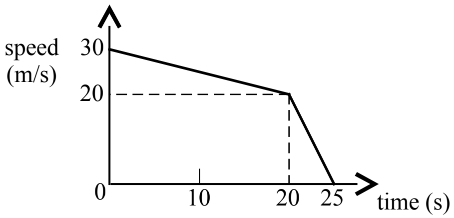
(c) By drawing a tangent, estimate the gradient of the curve at the point where $x = 3$.

(d) Estimate the area between the curve, the x-axis, $x = 2$ and $x = 5$.

(e) Using your graph, solve the equation $-5 = 4 - \dfrac{6}{x} + x$.

(f) Using your graph, solve the equation $10 - \dfrac{6}{x} - 3x = 0$.

4 The graph below shows how a car is decelerated uniformly from a speed of 30m/s to 20m/s in 20sec and then uniformly brought to rest after a further 5seconds.



Calculate: (a) the deceleration of the car for the first 20sec
(b) the total distance travelled in 25sec
(c) the average speed
(d) the speed of the car at 22sec

8 Inequations and Linear Programming

An inequation (inequality) is a mathematical statement that two algebraic expressions are not equal.

(ア) Basic Inequation

Inequation notation

The sign < means "less than".
The sign > means "greater than".
The sign ≤ means "less than or equal to"
The sign ≥ means "greater than or equal to".

e.g. $x > y$ means that x is greater than y.
$a \le b$ means that a is less than or equal to b.

Rules for inequations

If $a > b$, then

(i) $ax > bx$ and $\dfrac{a}{x} > \dfrac{b}{x}$ if x is **positive**.

(ii) $ax < bx$ and $\dfrac{a}{x} < \dfrac{b}{x}$ if x is **negative**.

> If an inequation is multiplied or divided by a negative number, the inequation sign must be reversed.

e.g. $6 > -2$

then $6 \times 2 > -2 \times 2$ i.e. $12 > -4$ and $\dfrac{6}{2} > \dfrac{-2}{2}$ i.e. $3 > -1$

but $6 \times (-2) < -2 \times (-2)$ i.e. $-12 < 4$ and $\dfrac{6}{-2} < \dfrac{-2}{-2}$ i.e. $-3 < 1$

(イ) Linear Inequations

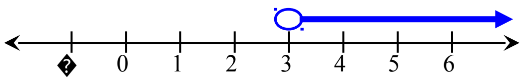
A linear inequation is an inequation involving an expression of the first degree.

Linear inequations in one variable: can be solved in a similar way to that used in solving linear equations.

Example 1 Solve: (a) $3x - 4 > 5$ (b) $2(1 - x) \geq 3(x + 4)$

(a) Move –4 to the right side. $3x > 5 + 4 = 9$ $\therefore x > 3$

This is illustrated on a number line. An open circle (○) means that the value at 3 is not included.

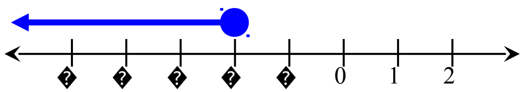


(b) First expand. $2 - 2x \geq 3x + 12$

Then collect like terms and simplify. $-2x - 3x \geq 12 - 2$ $-5x \geq 10$

Divide both sides by –5. (The sign must be reversed.) $x \leq -2$

This is illustrated on a number line. A shaded circle (●) means that the value at –2 is included.



Double inequation: represents two inequations.

e.g. $1 < x \leq 3$ x is greater than 1 and also less than or equal to 3.

Example 2 Solve $1 \leq 2x + 3 < 7$ and illustrate on a number line.

Take each inequation separately.

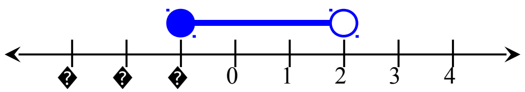
$1 \leq 2x + 3$ $\quad 2x + 3 < 7$

$1 - 3 \leq 2x$ $\quad 2x < 7 - 3$

$-2 \leq 2x$ $\quad 2x < 4$

$-1 \leq x$ $\quad x < 2$

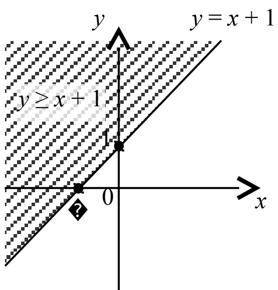
These solutions can be expressed by the double inequation $-1 \leq x < 2$.



Linear inequations in two variables: can be shown by the half planes on the coordinate plane.

Example 3 Show the solution of $y \geq x + 1$.

First draw the boundary line $y = x + 1$. Find the two points on the line.



The y-intercept is 1. ($\because$ 1 is constant)

At the x-intercept, $y = 0$ $0 = x + 1$ $\therefore x = -1$

Join these points, (0, 1) and (–1, 0)

To determine the half plane, take the origin (0, 0) as a test point. Substitute the coordinates of the origin into the inequation.

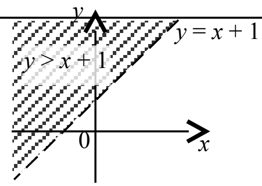
$0 \geq 0 + 1 = 1$ which is not true.

i.e. the inequation does not include the origin.

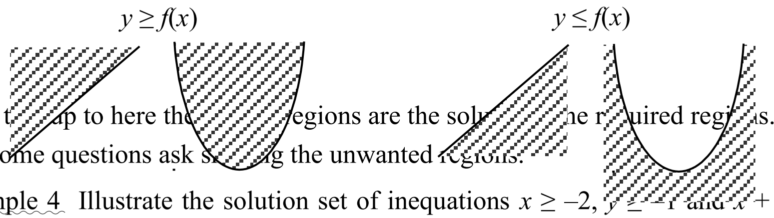
So the solution is above the line.

If the line does not pass through the origin, it should be used as the test point to determine the half plane.

If the inequation was $y > x + 1$, points on the boundary line are not included. To show this, a **broken (dotted) line** is used.



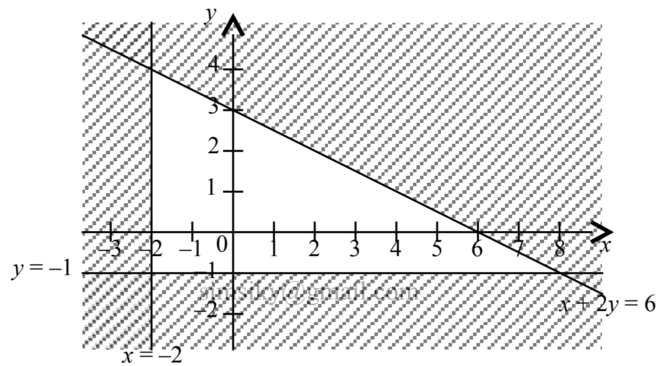
For any function, (i) if $y \geq f(x)$, the solution is **above** the line. (ii) if $y \leq f(x)$, the solution is **below** the line.



Note that up to here the regions are the solutions, the required regions.
But some questions ask shading the unwanted regions.

Example 4 Illustrate the solution set of inequations $x \geq -2$, $y \geq -1$ and $x + 2y \leq 6$ by shading the unwanted regions.

First draw the boundary lines one by one, then shade the unwanted regions.



Draw $x = -2$ and shade to the left ($x < -2$).
Next draw $y = -1$ and shade below the line ($y < -1$).

Then draw $x + 2y = 6 \Rightarrow y = -\frac{1}{2}x + 3$ and shade above the line $\left(y > -\frac{1}{2}x + 3\right)$.

$\left(\because x + 2y \le 6 \Rightarrow y \le -\frac{1}{2}x + 3\right)$

The solution set is shown by the unshaded region.

(ウ) Linear Programming

Linear programming is a method used to maximise or minimise a linear function subject expressed by linear inequations. Solutions are usually found by drawing graphs of inequations and looking for optimum values which satisfy the required conditions. Problems frequently relate to getting maximum profits for output and cost.

Example 5 A business man decides to buy two types of chickens, broilers and layers. Broilers cost K20 000 per each and layers cost K30 000 per each. He has K600 000. He decides that the total number of chickens should not be less than 20 and there should be at least 10 layers. He buys x broilers and y layers.

(a) Write down three inequalities which correspond to the above conditions.
(b) Illustrate these inequalities on a graph by shading the unwanted regions.
(c) He makes a profit of K5 000 on each broiler and K5 000 on each layer. Assuming he sells all his chickens, find how many chickens of each type he should buy to maximise his profit and calculate his profit.

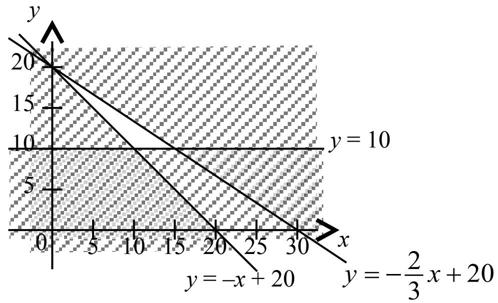
(a) Cost must be less than K600 000. $\Rightarrow 20\,000x + 30\,000y \le 600\,000$
Total number of chickens is not less than 20. $\Rightarrow x + y \ge 20$
There are at least 10 layers. $\Rightarrow y \ge 10$

(b) Rearrange the first inequation.

$20\,000x + 30\,000y \le 600\,000$ $2x + 3y \le 60$ $\therefore y \le -\frac{2}{3}x + 20$

Draw $y = -\frac{2}{3}x + 20$ and shade above the line $\left(y > -\frac{2}{3}x + 20\right)$.

Next rearrange the second inequation. $x + y \ge 20$ $\therefore y \ge -x + 20$
Draw $y = -x + 20$ and shade below the line ($y < -x + 20$).
Then draw the line $y = 10$ and shade below the line ($y < 10$).



(c) Total profit is $5\,000x + 5\,000y$.

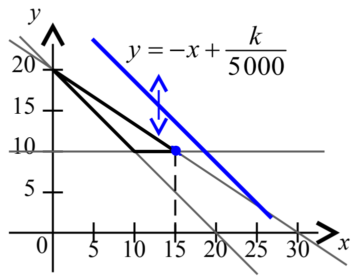
Let k represent the total profit. $5\,000x + 5\,000y = k$ $\therefore y = -x + \frac{k}{5000}$

This is the straight line in which the gradient is –1 and y-intercept is $\frac{k}{5000}$.

When the y-intercept is maximum, his profit is maximum.

The line $y = -x + \frac{k}{5000}$ can be drawn as shown below. When this line passes through the point (15, 10), the y-intercept becomes maximum.

Therefore $x = 15$ and $y = 10$.
(15 broilers and 10 layers)
Profit = 5 000 × 15 + 5 000 × 10
= K125 000



Exercise 8

1 Solve the inequations and show on a number line. (a) $2x > x - 1$ (b) $1 - x \ge x - 5$

(c) $\frac{x}{4}+1<6-x$ (d) $-1 < x + 2 \le 4$

2 Find the integer x such that $x + 2 < 7 < 2x + 1$

3 Given that $-2 \le x \le 3$ and $-6 \le y \le 4$, find: (a) the smallest possible value of $x + y$ (b) the largest possible value of $x - y$ (c) the value of y if $y^2 = 25$

4 Shade the solutions of the inequations. (a) $y > -x + 3$ (b) $3x - 2y \le 6$

5 Illustrate the solution sets of the inequations by shading the unwanted regions.

(a) $x \ge 0, y \ge 0, x + y < 4$ (b) $x > -2, y < 4, 2x + 3y \le 12$

(c) $y \le x + 2, y \le -2x + 6, 3y \ge -2x + 6$

6 A new tablet is being tested with two chemicals, acid A and acid B. Each milligram of acid A uses a volume of 1mm^3 and each milligram of acid B uses a volume of 2mm^3. It is given that each tablet contains x mg of acid A and y mg of acid B.

(a) Express each of the following conditions as an inequality involving x and y.

(i) The total mass of the chemicals must be at least 60mg.

(ii) The mass of acid B must be at least more than that of acid A.

(iii) The mass of acid B must not be more than twice that of acid A.

(iv) The volume of a tablet must not be more than 150mm^3.

(b) Show these four inequalities on a graph by shading the regions which are not required.

(c) Using your graph, find the mass of acid A and the mass of acid B in a tablet containing the greatest possible amount of acid B.

(d) Find the smallest possible volume of a tablet.

9 Proportion and Variation

(ア) Proportion

A proportion is a relationship of the equality of ratios between two pairs of quantities. The symbol $\propto$ means "is proportional to" or "varies as".

Direct proportion: is the relationship between quantities, such that both quantities increase or decrease in the same ratio.

e.g. If a speed(v) of a car is constant, the distance(x) travelled by the car is directly proportional to the time(t) taken as $x = vt$, i.e. $x \propto t$ (v is constant).

Inverse proportion: is the relationship between quantities, such that when one quantity increases the other decreases in the same ratio.

e.g. If a travel distance(x) is fixed, the time(t) taken is inversely proportional to the speed(v) of a car as $v = \frac{x}{t}$, i.e. $v \propto \frac{1}{t}$ (x is constant).

Example 1 A holiday is planned for a group of 20 children and food is bought to last for 15days. If the number of children were to increase to 25, how long would the same amount of food last?

20 children → 15 days The number of children is inversely proportional

25 children → x days to the number of days to last food.

simsiky@gmail.com

$20 \times 15 = 25 \times x \quad \therefore x = \dfrac{20 \times 15}{25} = 12$ days

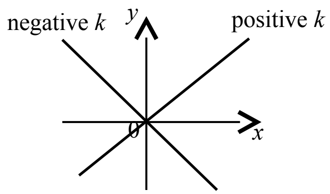
(イ) Variation

Variation requires you to find a constant in a formula that relates two or more variables in a ratio.

Direct variation: If y is directly proportional to x (y varies directly as x), this is written as $y \propto x$. Then this relationship can also be written as $y = kx$ where k is a constant (the constant of proportionality).

$y \propto x \quad \Rightarrow \quad y = kx$

The graph for direct variation is a straight line.



Example 2 y varies directly as x and $y = 6$ when $x = 2$. Find the value of y when $x = 5$

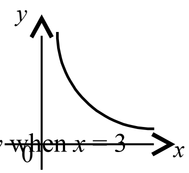
This relationship can be written as $y = kx$

Then $6 = k \times 2$ $k = 3$ $\therefore y = 3x$ So $y = 3 \times 5 = 15$

Inverse variation: If y is inversely proportional to x (y varies inversely as x), this is written as $y \propto \dfrac{1}{x}$. Then this relationship can also be written as $y = \dfrac{k}{x}$.

$y \propto \dfrac{1}{x} \quad \Rightarrow \quad y = \dfrac{k}{x}$ or $xy = k$

The graph for inverse variation is a curve (hyperbola).



Example 3 y varies inversely as x and $y = 4$ when $x = 6$. Find y when $x = 3$

This relationship can be written as $y = \dfrac{k}{x}$.

Then $k = xy = 6 \times 4 = 24$ $\therefore y = \dfrac{24}{x}$ So $y = \dfrac{24}{3} = 8$

Joint variation: is a variation in which one variable depends on two or more other variables.

e.g. The volume V of a cylinder is given by $V = \pi r^2 h$. V varies directly as the square of the radius (r) and directly as the height (h).

Example 4 y varies directly as z and inversely as x^2, and when $y = 6$, $z = 3$ $x = 2$. Find the values of y when $z = 6$ and $x = 4$.

This relationship can be written as $y = \dfrac{kz}{x^2}$.

Then $k = \dfrac{yx^2}{z} = \dfrac{6 \times 2^2}{3} = 8$ $\therefore y = \dfrac{8z}{x^2}$ So $y = \dfrac{8 \times 6}{4^2} = 3$

Exercise 9

1 It takes 6 people 12 hours to paint a house. If the work has to be completed in 8 hours how many people will be needed?

2 y varies directly as the square of x and $y = 8$ when $x = 2$. Find y when $x = 5$.

3 y varies inversely as $(x – 2)$ and $y = 2$ when $x = 5$. Find y when $x = 8$

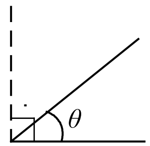
4 y varies directly as the square of x and inversely as z. $y = 5$ when $x = 1$ and $z = 3$. Find: (a) the equation (b) y when $x = 2$ and $z = 5$ (c) x when $y = 6$ and $z = 10$

10 Angles, Polygons and Bearings

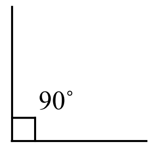
(ア) Angles

Types of angles

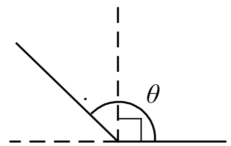
Acute angle $0° < \theta < 90°$



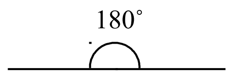
Right angle $\theta = 90°$



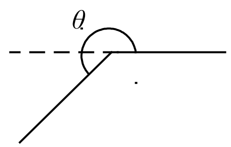
Obtuse angle $90° < \theta < 180°$



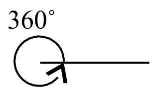
Straight angle $\theta = 180°$



Reflex angle $180° < \theta < 360°$

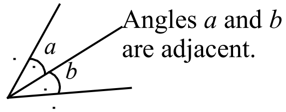


Full turn $\theta = 360°$



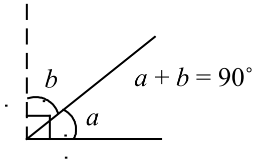
Related angles

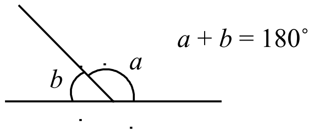
Adjacent angles: are angles which have a common vertex and a common line.



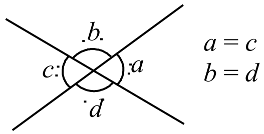
Complementary angles: are two angles which add up to 90˚.

Supplementary angles: are two angles which add up to 180˚.



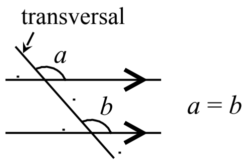


Vertically opposite angles: are the angles on opposite sides of the point where two straight lines cross. These pairs of angles are equal.

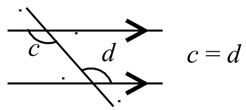


Angles associated with parallel lines:

Corresponding angles: are angles which lie on the same sides of the transversal and on the same relative sides of the parallel lines. Corresponding angles are equal.

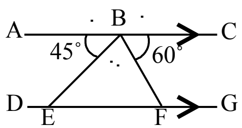


Alternate angles: are angles which lie on the opposite sides of the transversal and on opposite relative sides of the parallel lines. Alternate angles are equal.



Example 1 Find: (a) ∠BEG

(b) ∠BFG



(a) ∠BEG = ∠ABE (∵Alternate angles)

= 45˚

(b) ∠BFG = 180˚ – ∠BFD (∵Supplementary angles)

= 180˚ – ∠CBF (∵Alternate angles)

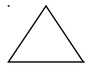
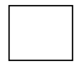
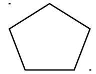
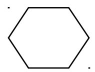
= 180˚ – 60˚ = 120˚

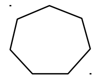
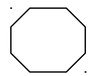
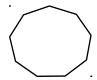
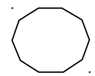
(イ) Polygons

A polygon is a plane shape enclosed by three or more straight lines.

A **regular** polygon is a polygon with all sides equal and all angles equal.

Types of polygons

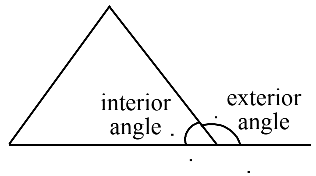
Number of sides	Name of polygon	Shape
3	Triangle	
4	Quadrilateral	
5	Pentagon	
6	Hexagon	

Number of sides	Name of polygon	Shape
7	Heptagon	
8	Octagon	
9	Nonagon	
10	Decagon	

Angles of a polygon

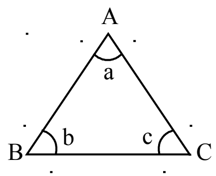
-An **interior angle** is the angle inside a polygon between two adjacent sides.

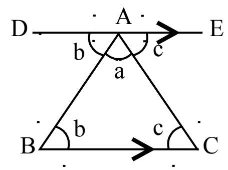
-An **exterior angle** is the angle between a side of a polygon and an adjacent side extended outwards.



Sum of the interior angles

e.g. Sum of the interior angles of a triangle





Draw a parallel line through A to side BC.

simsiky@gmail.com

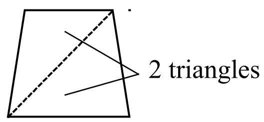
$\angle b = \angle DAB$ (∵Alternate angles) Similarly $\angle c = \angle EAC$

$\therefore a + b + c = 180^\circ$ (∵Straight angle)

e.g. Sum of the interior angles of a quadrilateral and a pentagon

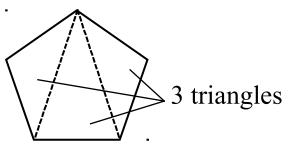
In a quadrilateral, if one diagonal is drawn, there are two triangles in it.

∴The sum of the interior angles of a quadrilateral $= 2 \times 180^\circ = 360^\circ$



In a pentagon, if two diagonals are drawn, there are three triangles in it.

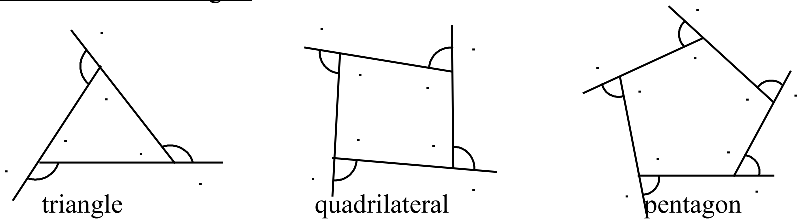
∴The sum of the interior angles of a pentagon $= 3 \times 180^\circ = 540^\circ$



Name of polygon	Number of sides	Number of triangles	Sum of interior angles
Triangle	3	1	$1 \times 180^\circ = 180^\circ$
Quadrilateral	4	2	$2 \times 180^\circ = 360^\circ$
Pentagon	5	3	$3 \times 180^\circ = 540^\circ$
⋮	⋮	⋮	⋮
n-sided polygon	n	$n - 2$	$(n - 2) \times 180^\circ$

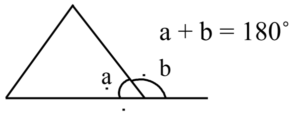
Sum of the interior angles of an n-sided polygon $= (n - 2) \times 180^\circ$

Sum of the exterior angles



An n-sided polygon has n exterior angles.

The sum of exterior angle and the interior angle next to it is 180° because of a straight angle.



$\therefore$Sum of exterior angles + sum of interior angles $= n \times 180^\circ$

Here sum of interior angles $= (n - 2) \times 180^\circ$

$\therefore$Sum of exterior angles $= n \times 180^\circ - (n - 2) \times 180^\circ = 360^\circ$

Sum of the exterior angles of any polygon = 360˚

Example 2 Find the size of each exterior and interior angles of a regular hexagon.

A hexagon has 6 exterior and 6 interior angles.

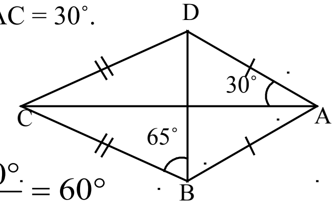
Each exterior angle $= \dfrac{360^\circ}{6} = 60^\circ$

Each interior angle $= 180^\circ - 60^\circ = 120^\circ$

or $\dfrac{(6-2)\times 180^\circ}{6} = 120^\circ$

Example 3 ABCD is a kite in which $\angle DBC = 65^\circ$ and $\angle DAC = 30^\circ$.

Find (a) $\angle DAB$ (b) $\angle ADB$ (c) $\angle ADC$ (d) $\angle DCB$



(a) △ABD is an isosceles triangle.

$\therefore \angle DAB = 2 \times \angle DAC = 2 \times 30^\circ = 60^\circ$

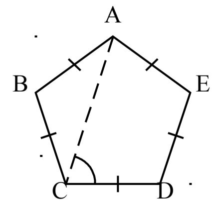
(b) $\angle ADB = \angle ABD = \dfrac{180^\circ - \angle DAB}{2} = \dfrac{180^\circ - 60^\circ}{2} = 60^\circ$

(c) △CBD is an isosceles triangle. $\therefore \angle CDB = \angle CBD = 65^\circ$

$\therefore \angle ADC = \angle ADB + \angle BDC = 60^\circ + 65^\circ = 125^\circ$

(d) $\angle DCB = 180^\circ - 2 \times \angle BDC = 180^\circ - 2 \times 65^\circ = 50^\circ$

Example 4 ABCDE is a regular pentagon. Find the size of $\angle ACD$.



Each interior angle $= \dfrac{(5-2)\times 180^\circ}{5} = 108^\circ$

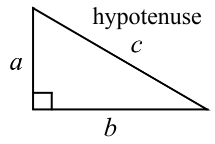
(or each exterior angle $= \dfrac{360^\circ}{5} = 72^\circ$ interior angle $= 180^\circ - 72^\circ$)

△ABC is an isosceles triangle. $\angle BCA = \dfrac{180^\circ - \angle ABC}{2} = \dfrac{180^\circ - 108^\circ}{2} = 36^\circ$

$\therefore \angle ACD = 108^\circ - 36^\circ = 72^\circ$

(ウ) Pythagoras Theorem

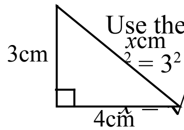
In any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.



$c^2 = \boxed{a^2 + b^2}$

This can be rearranged to: $c = \sqrt{a^2 + b^2}$

Example 5 Find the length of the side marked x.



Use the formula.

$x^2 = 3^2 + 4^2 = 9 + 16 = 25$

$x = \sqrt{25} = 5\text{cm}$

A set of three positive integers (a, b and c) which satisfy Pythagoras theorem ($c^2 = a^2 + b^2$) is called a Pythagoras triple.

There is an infinite number of Pythagoras triples. The most well known are:

3, 4, 5; 5, 12, 13; 7, 24, 25; 8, 15 ,17

Example 6 An equilateral triangle has sides of length 6cm. Find the height of the triangle.

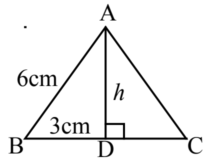
In an equilateral triangle, the lengths of all sides are equal.

In the diagram on the right, the height is AD.

BD is 3cm because of half of BC.

Use the formula. $6^2 = 3^2 + h^2$ $h^2 = 6^2 - 3^2 = 36 - 9 = 27$

$\therefore h = \sqrt{27} \approx 5.20\text{cm}$



Example 7 If A is the point (3, 1) and B is the point (5, 5)

(a) Plot the points A and B on a coordinate plane.

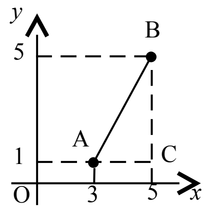
(b) Calculate the distance AB.

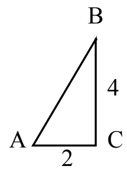
(a) The diagram is on the right.

(b) Let C represent a point (5, 1).

AC = 5 – 3 = 2, BC = 5 – 1 = 4

Use the formula.



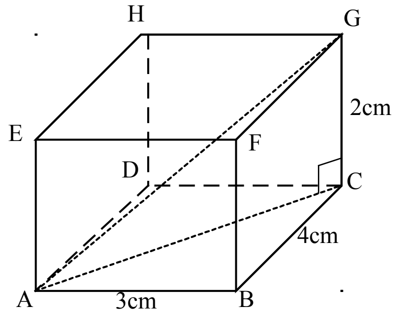


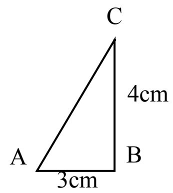
$AB = \sqrt{4^2 + 2^2} = \sqrt{20} \approx 4.47$

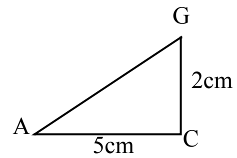
In general, the distance between points A (x_1, y_1) and B (x_2, y_2) can be calculated as follows:

$$\boxed{AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Example 8 ABCDEFGH is a cuboid in which AB = 3cm, BC = 4cm and CG = 2cm. Find the length of AG.







First find the length of AC.

$AC^2 = 3^2 + 4^2 = 25$ $\therefore AC = \sqrt{25} = 5\text{cm}$

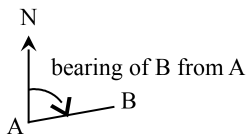
Then find the length of AG.

$AG^2 = 5^2 + 2^2 = 29$ $\therefore AG = \sqrt{29} \approx 5.39\text{cm}$

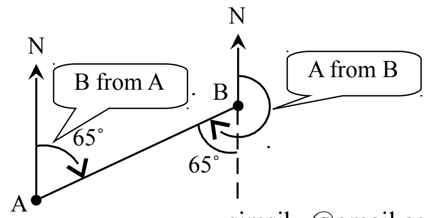
(エ) Bearings

A bearing is a direction measured in degrees, **clockwise from North**.

Bearings are written with **3 digits**. (e.g. 030˚, 120˚)



e.g.



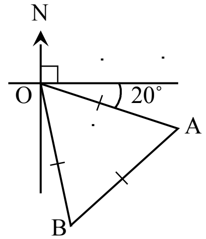
The bearing of B from A = 065˚

The bearing of A from B = 180˚ + 65˚

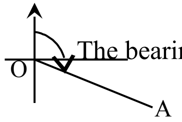
simsiky@gmail.com

= 245°

Example 9 Given that OAB is an equilateral triangle, find: (i) the bearing of A from O
(ii) the bearing of O from A
(iii) the bearing of B from O
(iv) the bearing of A from B

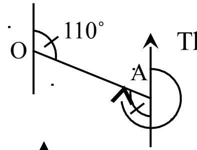


(i)



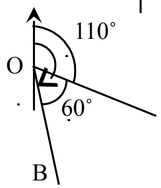
The bearing of A from O = 90° + 20°
= 110°

(ii)



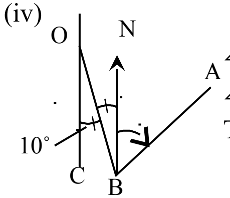
The bearing of O from A = 180° + 110°
= 290°

(iii)



Each interior angle of an equilateral triangle is 60°
The bearing of B from O = 110° + 60°
= 170°

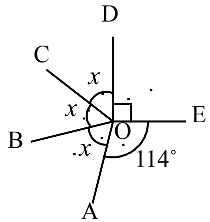
(iv)

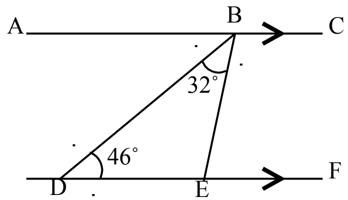


∠BOC = 180° – 170° = 10°
∠NBO = 10° (∵ Alternate angles)
The bearing of A from B = 060° – 010° = 050°

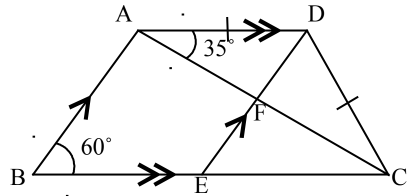
Exercise 10

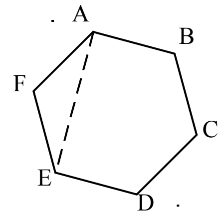
1 ∠AOB = ∠BOC = ∠COD = x and ∠AOE = 114°
Find x.



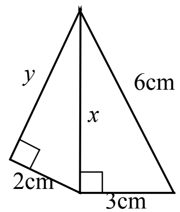


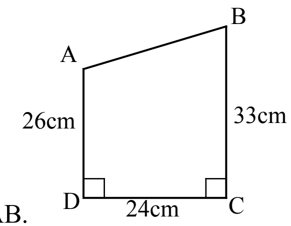
2 Find (a) ∠ABD (b) ∠EBC (c) ∠BEF
3 Find the sum of the interior angles of decagon.
4 Each interior angle of a regular polygon is 135°. Find the number of sides of this polygon
5 ABCD is a quadrilateral in which AB is parallel to DE, AD is parallel to BC, AD = DC, ∠ABE = 60° and ∠DAF = 35°. AFC is a straight line.
Calculate (a) ∠BED (b) ∠BAC (c) ∠CFD





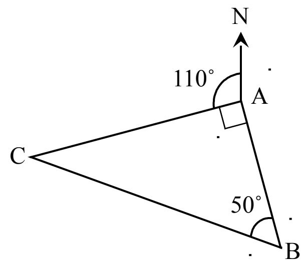
6 ABCDEF is a regular hexagon. Find the size of ∠BAE
7 Find the values of x and y.





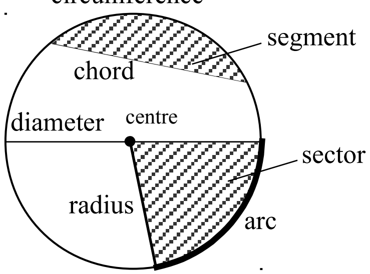
8 ABCD is a quadrilateral. Find the length of the side AB.
9 The length of the diagonal of a square is 4cm. Find the length of the sides of the square.
10 Find the distance between the points A(4, 6) and B(10, 14)

11 ABC is a right-angled triangle in which ∠NAC = 110° and ∠ABC = 50°. Find the bearing of:
(a) A from C
(b) B from A
(c) A from B
(d) C from B



11 Angle Properties in a Circle

(ア) Parts of a Circle



A **circumference** is the distance around the edge of a circle.
An **arc** is a part of the circumference.
A **radius** is a straight line from the centre of a circle to a point on the circumference.
A **diameter** is a straight line through the centre of a circle, joining two opposite points on the circumference. The diameter is twice the radius.
A **chord** is a straight line joining any two points on the circumference.
A **sector** is a part of a circle formed by two radii and an arc.
A **segment** is a part of a circle formed an arc and a chord.

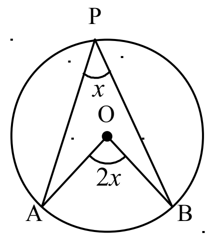
(イ) Angles in a Circle

Angles in a circle have some important properties.

Angles at the centre and on the circumference

The angle at the centre of a circle is twice the angle on the circumference of the circle (subtended by the same arc).

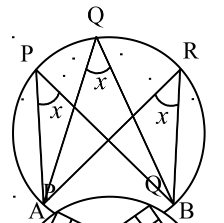
$$A\hat{O}B = 2 \times A\hat{P}B$$



Angles in the same segment

Angles in the same segment, subtended by the same arc, are equal.

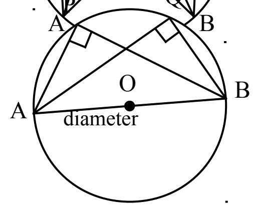
$$A\hat{P}B = A\hat{Q}B = A\hat{R}B$$



Angles in a semicircle

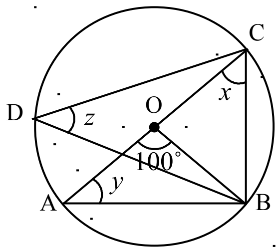
Any angle in a semicircle, subtended by a diameter, is a right angle.

$$A\hat{P}B = A\hat{Q}B = 90°$$



Example 1 In the diagram, AOC is the diameter.
Find x, y and z.

simsiky@gmail.com



$x = \frac{1}{2}\angle AOB = \frac{1}{2} \times 100° = 50°$

$y = \frac{180° - 100°}{2} = \frac{80°}{2} = 40°$

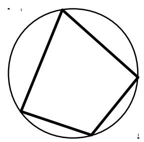
(∵△OAB is an isosceles triangle)

or $y = 180° – 50° – 90° = 40°$

($\angle ABC = 90°$ ∵ in a semicircle)

$z = y = 40°$ (∵same arc BC)

(ウ) Cyclic Quadrilaterals

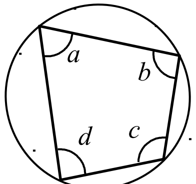


A cyclic quadrilateral is a quadrilateral which has all its vertices on the circumference of a circle.

Angles within a cyclic quadrilateral have some important properties.

Opposite angles of a cyclic quadrilateral

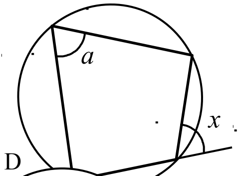
Opposite angles of a cyclic quadrilateral add up to 180°.



$a + c = 180°$ and $b + d = 180°$

Exterior and opposite interior angles of a cyclic quadrilateral

The exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.



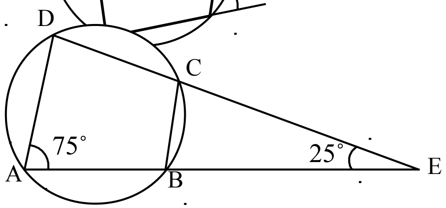
$a = x$

Example 2 ∠BAD = 75° and ∠BEC = 25°.

Calculate: (a) ∠BCD

(b) ∠ADC

(c) ∠CBE



(a) ∠BCD = 180° – ∠BAD (∵opposite angles)

= 180° – 75° = 105°

(b) ∠ADC = 180° – ∠BAD – ∠AEC (∵△ADE)

= 180° – 75° – 25° = 80°

(c) ∠CBE = ∠ADC (∵exterior and opposite interior angles)

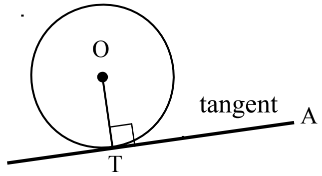
= 80°

(エ) Tangents

A tangent to a circle is a straight line which touches the circle at a point.

Radius at the point of contact

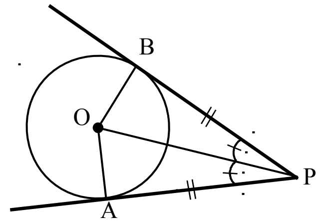
A tangent to a circle is always perpendicular to the radius at the point of contact.



$O\hat{T}A = 90°$

Two tangents from an outside point

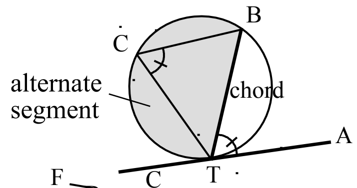
Two tangents to a circle from an outside point are equal in length and the line joining the centre of the circle and the point bisects the angle between the two tangents.



AP = BP and $A\hat{P}O = B\hat{P}O$

Alternate segment

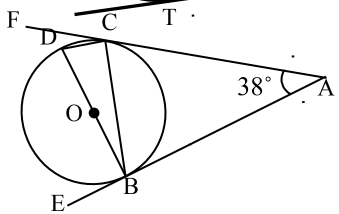
The angle formed between a tangent and a chord from the point of contact is equal to any angle subtended by the chord in the alternate segment.



$A\hat{T}B = B\hat{C}T$

Example 3 In the diagram, ABE and ACF are the tangents to the circle at B and C respectively. BD is a diameter and ∠BAC = 38°. Calculate:

(a) ∠CBA (b) ∠DBA (c) ∠BDC



(a) $\angle CBA = \frac{180° - 38°}{2} = 71°$ (∵isosceles triangle)

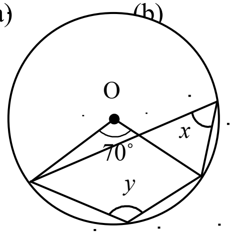
(b) ∠DBA = 90° (∵angle between tangent and radius)

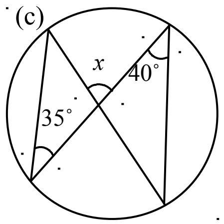
(c) ∠BDC = ∠CBA = 71° (∵angle in an alternate segment)

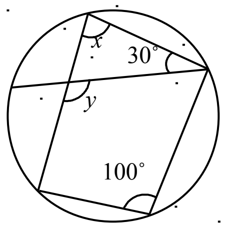
Exercise 11

12 Find the size of each angle marked by letters. O is the centre of the circle.

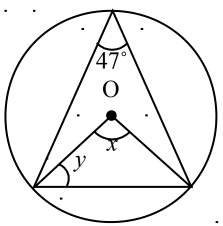
(a) (b) (c)

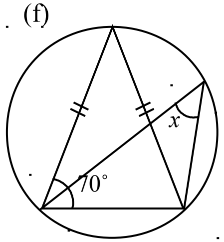


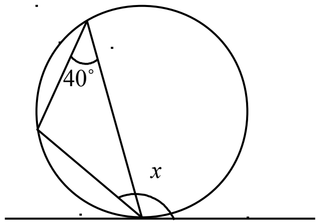




(d) (e) (f)

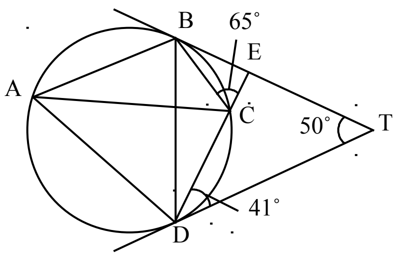






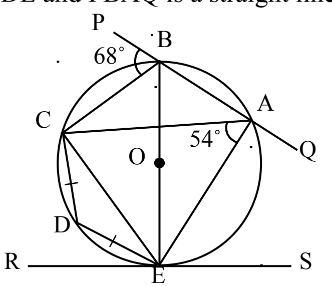
13 In the diagram below, TB and TD are tangents to the circle through A, B, C and D. DC produced meets TB at E. Given that ∠BCE = 65°, ∠DTE = 50° and ∠CDT= 41°, calculate:

(a) ∠BDC
(b) ∠CEB
(c) ∠BAD
(d) ∠DBC



14 In the diagram below, O is the centre of the circle with diameter BE. A, B, C, D and E are points on the circle. RES is a tangent to the circle. Given that ∠CBP = 68°, ∠CAE= 54°, CD = DE and PBAQ is a straight line, calculate:

(a) ∠REC
(b) ∠BEC
(c) ∠SEA
(d) ∠CED



15 Symmetry

A shape has symmetry when parts of that shape fit exactly onto other parts of the same shape under certain rules of movement.

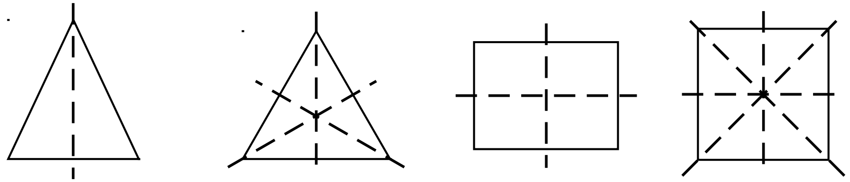
A plane shape can have two types of symmetry, line symmetry and rotational symmetry.

simsiky@gmail.com

(ア) Line Symmetry

Line symmetry is the symmetry in which a shape can be fold along a line so that one half of the shape fits exactly onto the other half.

e.g. symmetrical shapes (lines of symmetry shown in broken lines)



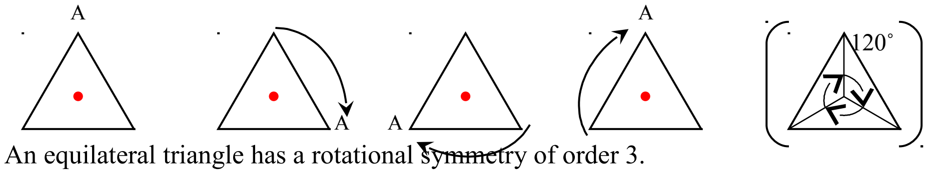
isosceles triangle   equilateral triangle   rectangle   square

(イ) Rotational Symmetry

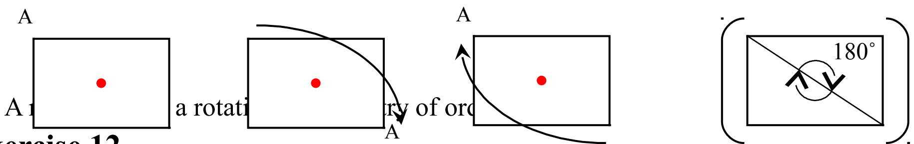
Rotational symmetry is the symmetry in which a shape can be turned about a fixed point and fit exactly onto itself.

Order of rotational symmetry: is the number of times that a shape can be turned to fit exactly onto itself within 360°. Every shape has an order of rotational symmetry of at least 1. $\frac{360^\circ}{\text{order of rotational symmetry}}$ represents the angle of rotation.

e.g. (The letter A is used only to show the positions of the shape as it turns.)



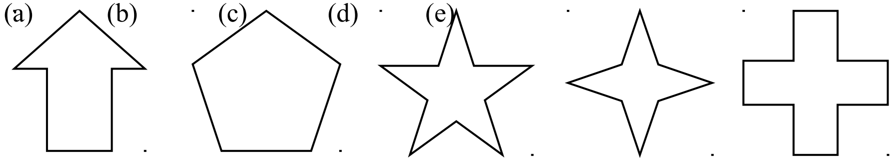
An equilateral triangle has a rotational symmetry of order 3.



Exercise 12

16 Copy the figure below and show its lines of symmetry.

(a) (b) (c) (d) (e)

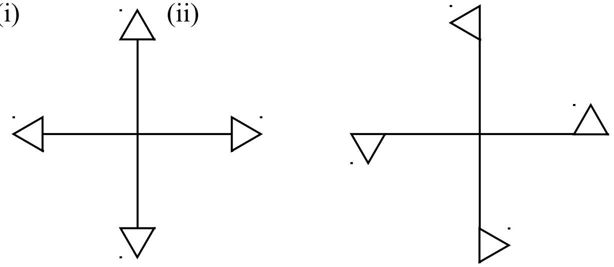


17 State the number of lines of symmetry and the order of rotational symmetry for each of the following letters: A D E F H M S T X Y

18 State the number of lines of symmetry and the order of rotational symmetry of: (a) a regular hexagon, (b) a regular octagon, (c) a parallelogram, (d) a kite, (e) a rhombus, (f) an isosceles triangle, (g) a circle.

19 For each of the following figures, state (a) the number of lines of symmetry and (b) the order of rotational symmetry.

(i) (ii)



20 Similarity

Geometrical figures are **similar** if they are the same in shape but different in size, such as those produced by enlargement.

(They are said to be **congruent** if they are the same in shape and size.)

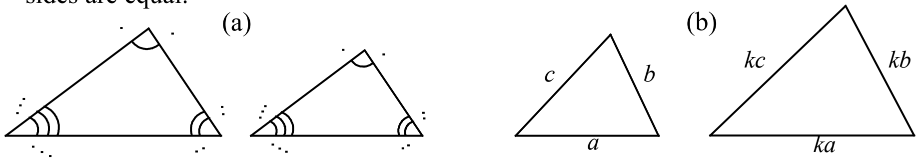
(ア) Similar Triangles

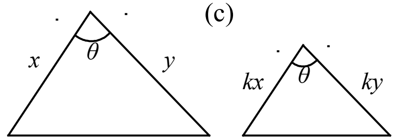
Two given triangles are similar if one of the following conditions can be shown.

(a) the corresponding angles are equal.

(b) the corresponding sides are in the same ratio.

(c) two pairs of corresponding sides are in the same ratio and the angles between these sides are equal.



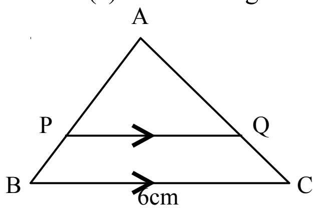


The ratio of corresponding sides is called the **scale factor** (k) of the enlargement.

Example 1 In △ABC, PQ is parallel to BC, AP = 2PB and BC = 6cm.

(a) Show that △APQ is similar to △ABC.

(b) Find the length of PQ,



(a) ∠A is common.

∠ABC = ∠APQ and ∠ACB = ∠AQP (∵corresponding angles)

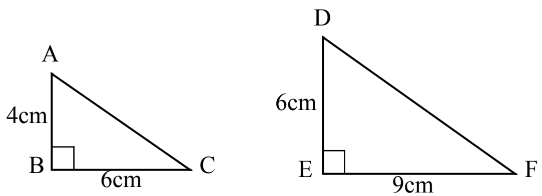
The corresponding angles are equal. Therefore △APQ is similar to △ABC.

(b) $\dfrac{AP}{AB} = \dfrac{AQ}{AC} = \dfrac{PQ}{BC}$

Now $\dfrac{AP}{PB} = \dfrac{2}{1}$ so $\dfrac{AP}{AB} = \dfrac{2}{3} = \dfrac{PQ}{BC}$

$\therefore PQ = \dfrac{2}{3}BC = \dfrac{2}{3} \times 6 = 4\text{ cm}$

(イ) Areas of Similar Figures



$\dfrac{DE}{AB} = \dfrac{EF}{BC} = \dfrac{3}{2}$ and ∠ABC = ∠DEF.

Therefore △ABC is similar to △DEF.

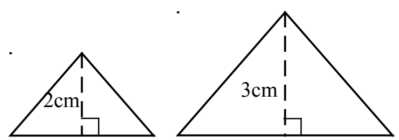
$$\text{Area of } \Delta ABC = \frac{1}{2} \times 6 \times 4 = 12\text{ cm}^2$$

$$\text{Area of } \Delta DEF = \frac{1}{2} \times 9 \times 6 = 27\text{ cm}^2$$

$$\frac{\text{Area of } \Delta DEF}{\text{Area of } \Delta ABC} = \frac{27}{12} = \frac{9}{4} = \left(\frac{3}{2}\right)^2$$

> If the scale factor of two similar figures is k, the ratio of their areas is k^2.
> (If the ratio of the sides of two similar figures is $a : b$, the ratio of their areas is $a^2 : b^2$.)

Example 2 The given triangles are similar. If the area of the smaller triangle is 8cm^2, find the area of the larger triangle.



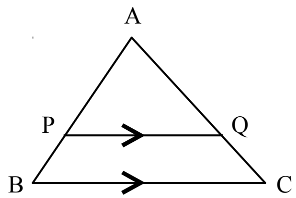
Ratio of sides = 2 : 3

So ratio of areas = $2^2 : 3^2 = 4 : 9$

Let x represent the area of the larger triangle.

$4 : 9 = 8 : x \quad 4x = 9 \times 8 \quad \therefore x = \dfrac{9 \times 8}{4} = 18\text{ cm}^2$

Example 3 In △ABC, PQ is parallel to BC. If the area of the trapezium PBCQ is $\dfrac{16}{25}$ of △ABC, find the ratio of AP : PB.



△APQ is similar to △ABC.

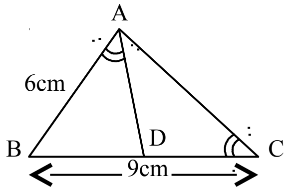
$$\text{Area of } \Delta APQ = \left(1 - \frac{16}{25}\right) \text{of } \Delta ABC = \frac{9}{25} \text{ of } \Delta ABC$$

$$\frac{\text{Area of } \Delta APQ}{\text{Area of } \Delta ABC} = \frac{9}{25} \text{ so } \frac{AP}{AB} = \sqrt{\frac{9}{25}} = \frac{3}{5} \quad \therefore \frac{AP}{PB} = \frac{3}{2}$$

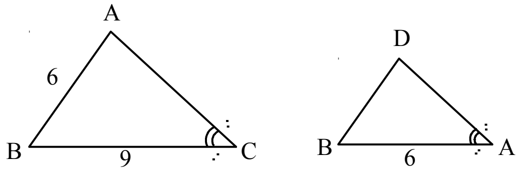
simsiky@gmail.com

∴AP : PB = 3 :2

Example 4 In △ABC, D is a point on BC such that ∠BAD = ∠ACB. Given that AB = 6cm, BC = 9cm and the area of △ABD is 12cm². calculate:
(a) the length of BD
(b) the area of △ABC



(a) ∠B is common and ∠BAD = ∠ACB. Therefore △DBA is similar to △ABC.



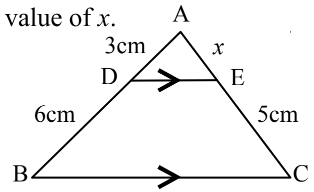
$$\frac{DB}{AB}=\frac{BA}{BC} \quad \frac{DB}{6}=\frac{6}{9} \quad \therefore DB=\frac{2}{3}\times 6=4\text{ cm}$$

(b)
$$BC=\frac{3}{2}BA$$

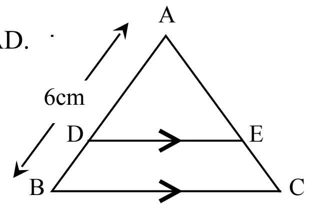
$$\therefore \text{Area of } \Delta ABC=\left(\frac{3}{2}\right)^2\times \text{area of } \Delta ABD=\frac{9}{4}\times 12=27\text{ cm}^2$$

Exercise 13

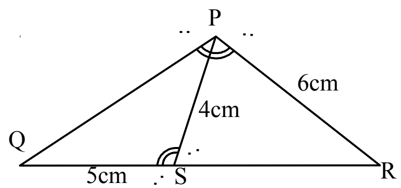
21 Calculate the value of x.



1 In △ABC, AB = 6cm and the area of △ADE is $\frac{4}{9}$ of △ABC. Calculate the length of AD.



2 In △PQR, S is a point on QR such that ∠QSP = ∠QPR. Given that PS = 4cm, QS = 5cm, PR = 6cm and the area of △PQR is 18cm², calculate: (a) the length of PQ (b) the area of △PQS



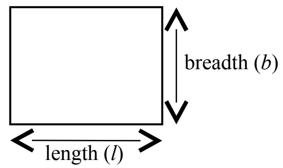
22 Mensuration

Mensuration means the measurement and calculation of perimeters, areas and volumes. Some basic formulas are given as following. Those should be memorised.

(ア) Perimeters and Areas of Plane Figures

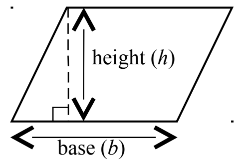
P = perimeter, A = area

Rectangle



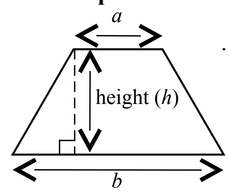
$P = 2(l + b)$
$A = lb$

Parallelogram



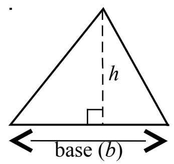
$A = bh$

Trapezium



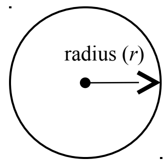
$A=\frac{1}{2}(a+b)h$

Triangle



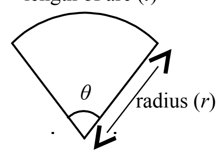
$A=\frac{1}{2}bh$

Circle



P (circumference)
$= 2\pi r = \pi d$ (diameter)
$A = \pi r^2$

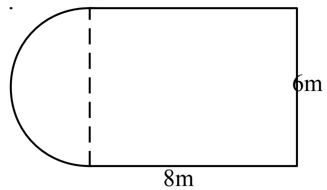
Sector of circle



$l=\frac{\theta}{360°}\times 2\pi r$

$A=\frac{\theta}{360°}\times \pi r^2$
$=\frac{1}{2}lr$

Example 1 The diagram on the right shows a flower bed consisting of a rectangle and a semicircle. Taking π = 3.14, find: (a) the perimeter
(b) the area



(a) Perimeter = sum of three sides of the rectangle
+ half circumference

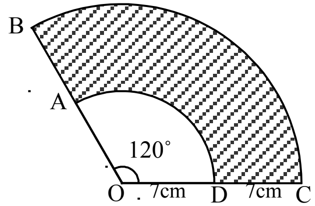
$= 8+6+8+\frac{1}{2}\times\pi d = 22+\frac{1}{2}\times 3.14\times 6$

$= 22 + 9.42 = 31.42$ m

(b) Area = area of rectangle + area of semicircle

$= 8\times 6+\frac{1}{2}\times\pi r^2 = 48+\frac{1}{2}\times 3.14\times 3^2$

$= 48 + 14.13 = 62.13$ m^2



Example 2 Take $\pi = \frac{22}{7}$. In the diagram on the right, find: (a) the shaded area

(b) the perimeter of this area

(a) Shaded area = area of sector OBC – area of sector OAD

area of sector OBC = $\frac{120°}{360°}\times\pi\times 14^2$ area of sector OBC = $\frac{120°}{360°}\times\pi\times 7^2$

Shaded area = $\frac{120°}{360°}\times\pi\times\left(14^2-7^2\right)$

$=\frac{1}{3}\times\frac{22}{7}\times(14+7)(14-7)$ $\Leftarrow a^2 - b^2 = (a + b)(a - b)$

$= 154$ cm^2

(b) Perimeter = sum of lengths of AB, arc BC, CD and arc DA

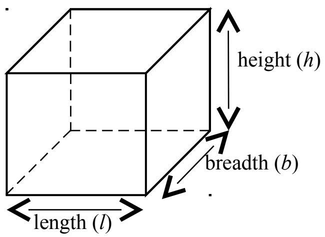
$=7+\frac{120°}{360°}\times 2\times\pi\times 14+7+\frac{120°}{360°}\times 2\times\pi\times 7$

$= 14+\frac{1}{3}\times 2\times\frac{22}{7}\times(14+7) = 14 + 44 = 58$ cm

(イ) Surface Areas and Volumes of Three-Dimensional Figures

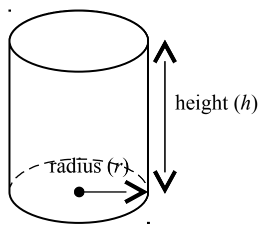
A = surface area, V = volume

Cuboid



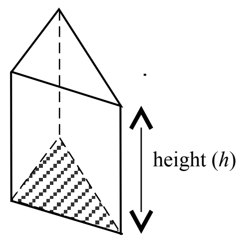
$A = 2lb + 2hb + 2hl$
$V = lbh$

Cylinder



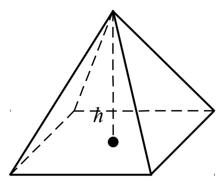
$A = 2\pi rh + 2\pi r^2$
$V = \pi r^2 h$

Prism



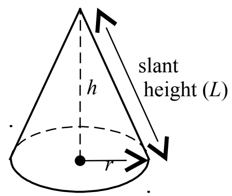
V = (area of base) ラ h

Pyramid



$V = \frac{1}{3}\times(\text{area of base})\times h$

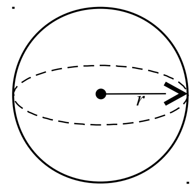
Cone



$A = \pi rL + \pi r^2$

$V = \frac{1}{3}\pi r^2 h$

Sphere



$A = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

Volume of regular prisms

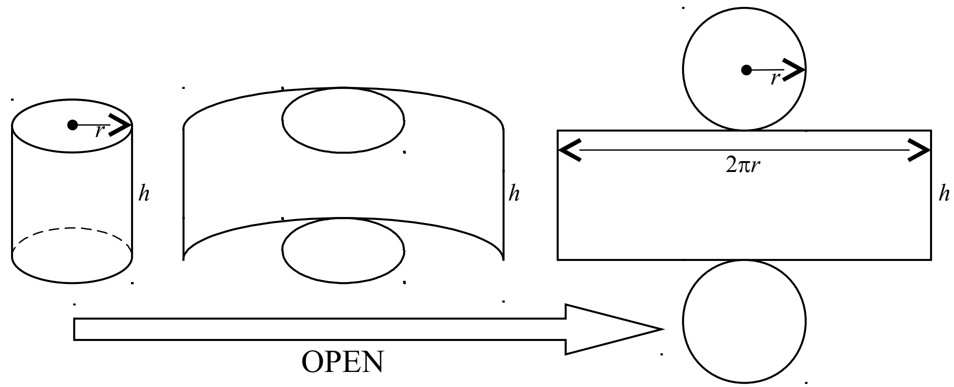
In any regular prism (including cuboids and cylinders),

Volume = area of base (or cross-section) × height

In a cuboid, area of base = area of rectangle = lb. So $V = lbh$

In a cylinder, area of base = area of circle = πr^2. So $V = \pi r^2 h$

Surface area of a cylinder: consists of two circles and a curved face, and can be opened as shown below. You will see that the curved face is a rectangle. The length of the rectangle is equal to the circle's circumference.

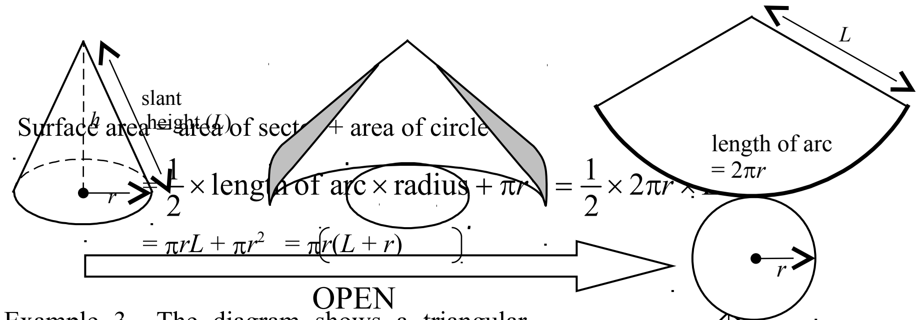


Surface area = area of rectangle + 2 × area of circle

$= 2\pi r \times h + 2\times \pi r^2$

$= 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$

Surface area of a cone: consists of a circle and a curved face, and can be opened as shown below. You will see that the curved face is a sector. The length of the arc is equal to the circle's circumference.



Surface area = area of sector + area of circle

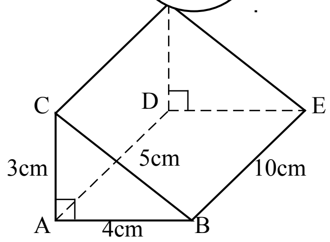
$= \frac{1}{2} \times \text{length of arc} \times \text{radius} + \pi r^2 = \frac{1}{2} \times 2\pi r \times$ ...

$= \pi rL + \pi r^2 = \pi r(L + r)$

Example 3 The diagram shows a triangular prism ABCDEF such that AB = 4cm, BC = 5cm, CA = 3cm and BE = 10cm. Calculate:

(a) the surface area of the prism

(b) the volume of the prism



(a) Surface area = area of two triangles + area of three rectangles

Area of $\triangle$ABC $= \frac{1}{2} \times 4 \times 3 = 6\ \text{cm}^2$

Area of rectangle ABED = 4 × 10 = 40 cm^2

Area of rectangle CBEF = 5 × 10 = 50 cm^2

Area of rectangle CADF = 3 × 10 = 30 cm^2

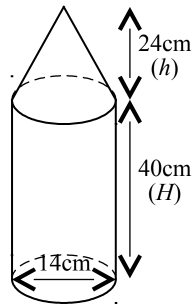
∴Surface area = 2 × 6 + 40 + 50 + 30 = 132 cm^2

(b) Volume = area of base × length of prism

= area of $\triangle$ABC × length of prism

= 6 × 10 = 60 cm^3

Example 4 An object consists of a cylinder and a cone with the same diameter 14cm. The height (H) of the cylinder is 40cm and the height (h) of the cone is 24cm. Taking $\pi = \frac{22}{7}$, calculate:

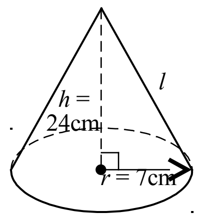


(a) the slant length (l) of the cone

(b) the total surface area of the object

(c) the total volume of the object

(a) By Pythagoras theorem,



$l^2 = r^2 + h^2$

$l = \sqrt{r^2 + h^2} = \sqrt{7^2 + 24^2}$

$= \sqrt{49 + 576} = \sqrt{625} = 25\ \text{cm}$

(b) Total surface area = area of circle + area of curved face of cylinder + area of curved face of cone

$= \pi r^2 + 2\pi rH + \pi rl = \pi r(r + 2H + l)$

$= \frac{22}{7} \times 7 \times (7 + 2 \times 40 + 25) = 22 \times 112 = 2464\ \text{cm}^2$

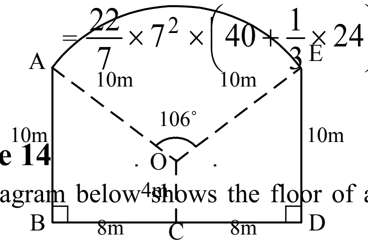
(c) Total volume = volume of cylinder + volume of cone

$= \pi r^2 H + \frac{1}{3}\pi r^2 h = \pi r^2\left(H + \frac{1}{3}h\right)$

$= \frac{22}{7} \times 7^2 \times \left(40 + \frac{1}{3} \times 24\right) = 22 \times 7 \times 48 = 7392\ \text{cm}^3$

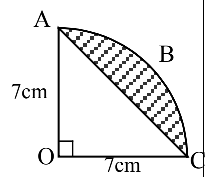
Exercise 14

1 The diagram below shows the floor of a room. OABC and OCDE are trapeziums.

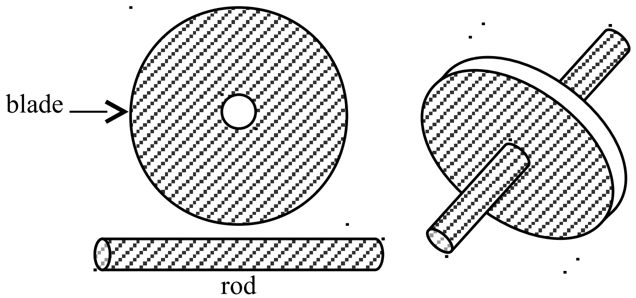


OAE is a sector of a circle with centre O, and ∠AOE = 106°. Taking π = 3.142, calculate: (a) the perimeter (b) the area

2 OABC is a quadrant of radius 7cm. Taking $\pi = \frac{22}{7}$, calculate he shaded area.



3 Take π to be $\frac{22}{7}$. The diagram below shows a cross section of a metallic blade that can be fitted to a solid cylindrical rod.



(a) Given that the rod has a diameter of 14mm and is 40cm long, calculate the volume of the rod in cm^3.

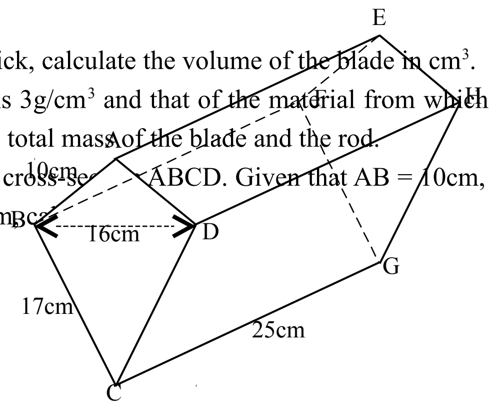
(b) The blade has a diameter of 14cm and the hole at the centre of the blade is such that it allows the rod to exactly fit in it. Calculate the surface area of one face of the blade in cm^2.

(c) Given that the blade is 20mm thick, calculate the volume of the blade in cm^3.

(d) The density of the metal blade is 3g/cm^3 and that of the material from which the rod is made is 6g/cm^3. Calculate the total mass of the blade and the rod.

4 ABCDEFGH is a prism of uniform cross-section ABCD. Given that AB = 10cm, BC = 15cm, BD = 17cm and CG = 25cm, calculate

(a) the length of AC



(b) the area of quadrilateral ABCD

(c) the volume of the prism

23 Geometrical Constructions and Loci

Never rub out arcs or construction lines because these show how you obtained your result.

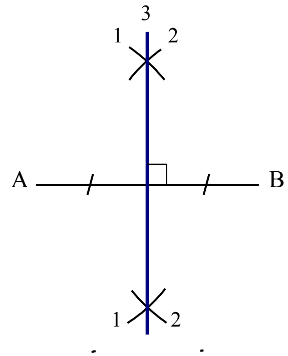
(ア) Basic Constructions

To bisect a line

Given a line AB.

1. With the centre at A and a radius of more than half the length of AB, draw an arc on each side.
2. With the centre at B and the same radius, draw another arc on each side to cut the first arc.
3. Join both points of intersection of the arcs.

The bisector is perpendicular to AB and any point on the bisector is equidistant from A and B.



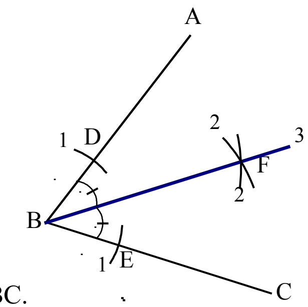
To bisect an angle

Given an angle ABC.

1. With the centre at B and a suitable radius, draw an arc to cut BA at D and BC at E.
2. With the centre at D and E, and a suitable and the same radius, draw arcs to intersect at F.
3. Draw a line from B through F.

∠ABF = ∠CBF

Any point on the bisector is equidistant from the lines AB and BC.

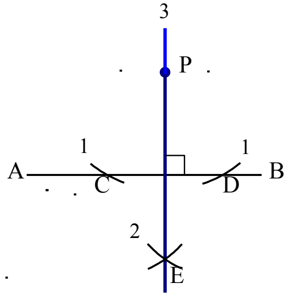


simsiky@gmail.com

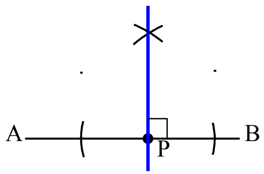
To construct a perpendicular line through a particular point outside a given line

Given a line AB and a point P.

1. With the centre at P, draw an arc to cut AB at C and D.
2. With the centre at C and D, and a radius of more than half the length of CD, draw arcs to intersect at E.
3. Draw a line from P through E.



This construction can be used even if a given point is on the line AB.

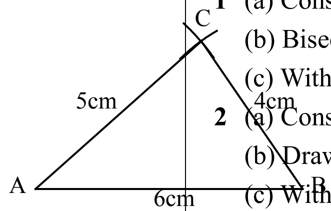


(イ) Constructing triangles

Given three sides

e.g. △ABC in which AB = 6cm, BC = 4cm and AC = 5cm.

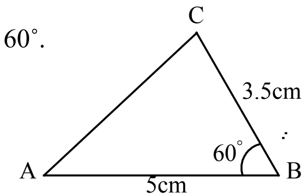
1. Draw a straight line AB of length 6cm.
2. With A as the centre and radius of 5cm, draw an arc.
3. With B as the centre and radius of 4cm, draw an arc to cut the first arc The intersection is C.
4. Join AC and BC.



Given two sides and the included angle

e.g. △ABC in which AB = 5cm, BC = 3.5cm and ∠ABC = 60˚.

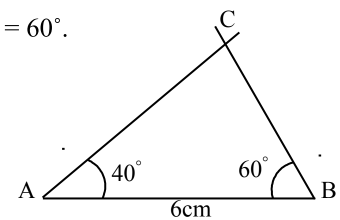
1. Draw a straight line AB of length 5cm.
2. Measure the included angle 60˚ at B with a protractor, mark it and draw a straight line BC of length 3.5cm.
3. Complete the triangle by joining AC.



Given a side and two angles

e.g. △ABC in which AB = 6cm, ∠BAC = 40˚ and ∠ABC = 60˚.

1. Draw a straight line AB of length 6cm.
2. Measure angle 40˚ at A and angle 60˚ at B with a protractor and draw AC and BC to meet at C.



(ウ) Loci

'Loci' is the plural of 'locus'. A locus is a set of points which satisfy one or more given conditions.

Example 1 The points A and B are 4cm apart. Find the set of points that are less than 3cm from A but are closer to B than A.

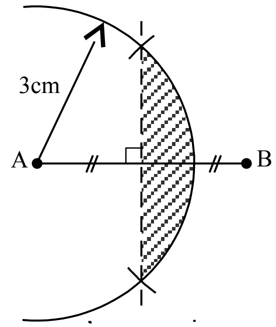
First, to find the points which are less than 3cm from A, draw an arc of radius 3cm.

Then, to find the points which are closer to B than A, draw the perpendicular bisector of AB.

Here the perpendicular bisector is a dotted line.

Points on it are not included because they are the same distance from A and B.

Finally, shade the region which satisfies the conditions.



Exercise 15

1 (a) Construct △ABC in which AB = 9cm, BC = 7cm and 5cm.

(b) Bisect ∠ABC and ∠CAB, and mark the intersection as O.

(c) With O as the centre and radius to touch AB, draw a circle.

2 (a) Construct △ABC in which AB = 6cm, ∠ABC = 40˚ and ∠CAB = 55˚.

(b) Draw perpendicular bisectors of all the sides and mark the intersection as O.

(c) With O as the centre and radius of the length of OA, draw a circle.

3 (a) Construct a triangle ABC in which AB = 9cm, BC = 7cm and ∠ABC = 40˚

(b) On the same diagram, draw the locus of points within the triangle which are

(i) 6cm from B.

(ii) equidistant from AC and AB.

(c) P is a point inside △ABC such that it is 6cm from B, and equidistant from AB and AC. Label the point P.

(d) A point Q lies inside △ABC such that its distance from B is less than 6cm and it is nearer to AC than to AB. Shade the region in which Q must lie.

24 Trigonometry

Trigonometry deals with the relations between the sides and angles of a triangle.

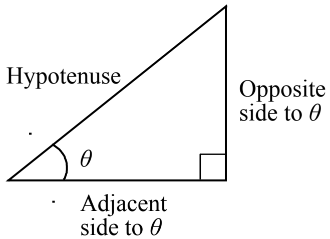
(ア) Trigonometric Ratios

Ratio of sides of a right-angled triangle: There are three ratios in a right-angled triangle, namely **sine**(sin), **cosine**(cos) and **tangent**(tan), defined as follows.

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \text{ (SOH)}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \text{ (CAH)}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} \text{ (TOA)}$$

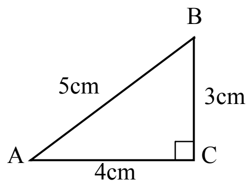


SOH CAH TOA is useful to memorise the three ratios.

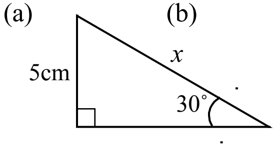
e.g. In the triangle on the right,

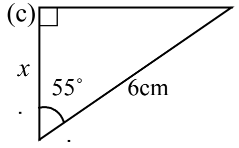
$$\sin A = \frac{BC}{AB} = \frac{3}{5} \quad \cos A = \frac{AC}{AB} = \frac{4}{5} \quad \tan A = \frac{BC}{AC} = \frac{3}{4}$$

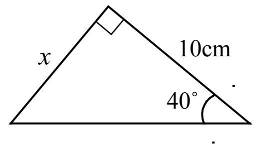
$$\sin B = \frac{AC}{AB} = \frac{4}{5} \quad \cos B = \frac{BC}{AB} = \frac{3}{5} \quad \tan B = \frac{AC}{BC} = \frac{4}{3}$$



Example 1 Calculate the value of x in each of the following.







(a) $\sin 30° = \dfrac{5}{x} \quad \therefore x = \dfrac{5}{\sin 30°} = \dfrac{5}{0.5} = 10 \text{ cm}$

(b)

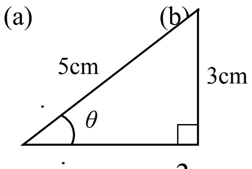
$$\cos 55° = \frac{x}{6} \quad \therefore x = 6 \times \cos 55° = 6 \times 0.5736 = 3.4416 = 3.44 \text{ cm (to 3sf)}$$

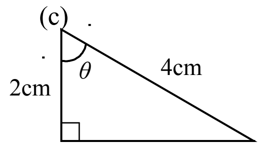
(c)

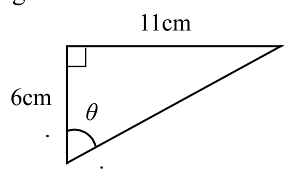
$$\tan 40° = \frac{x}{10} \quad \therefore x = 10 \times \tan 40° = 10 \times 0.8391 = 8.391 = 8.39 \text{ cm (to 3sf)}$$

The sine, cosine and tangent ratios can also be used to find the size of an unknown angle in a right-angled triangle. $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ give the angle which has a sine, cosine and tangent of θ and are the inverse of the sine, cosine and tangent respectively.

Example 2 Calculate the value of θ in each of the following.





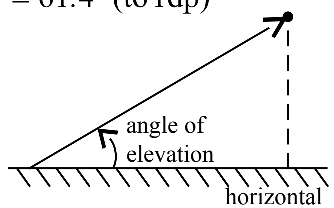


(a) $\sin\theta = \dfrac{3}{5} \quad \therefore \theta = \sin^{-1}\dfrac{3}{5} = 36.86° = 36.9° \text{ (to 1dp)}$

(b) $\cos\theta = \dfrac{2}{4} = \dfrac{1}{2} \quad \therefore \theta = \cos^{-1}\dfrac{1}{2} = 60°$

(c) $\tan\theta = \dfrac{11}{6} \quad \therefore \theta = \tan^{-1}\dfrac{11}{6} = 61.38° = 61.4° \text{ (to 1dp)}$

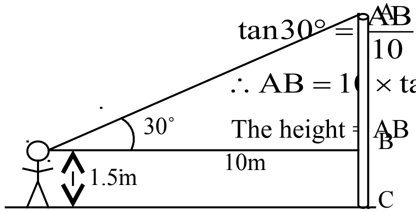
Angles of elevation: are measured upward from the horizontal when looking up to a point.



Example 3 The angle of elevation of the top

simsiky@gmail.com

of a pole from the eye-level of the boy standing 10m away from the foot of the pole is 30˚.
If the height of the boy up to eye-level is 1.5m, find the height of the pole.



$\tan 30° = \frac{AB}{10}$

$\therefore AB = 10 \times \tan 30° = 5.77\text{ m}$

The height = AB + BC

= 5.77 + 1.5

= 7.27 = 7.3m (to 1dp)

Identity: The equation $\boxed{\sin^2\theta + \cos^2\theta = 1}$ is true for all values of θ and it is called the identity.

Example 4 Given that $\sin\theta = 0.6$ and $0˚ \le \theta \le 90˚$, find the value of $\cos\theta$.

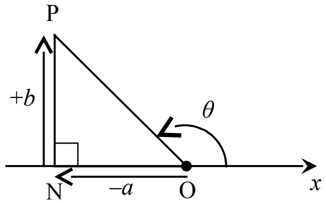
$\cos^2\theta = 1 - \sin^2\theta = 1 - 0.6^2 = 1 - 0.36 = 0.64$

$\therefore \cos\theta = \sqrt{0.64} = 0.8$ (when $0˚ \le \theta \le 90˚$, $\cos\theta$ is positive.)

(イ) Non-right-angled Triangles

Even if a triangle does not contain a right angle, sine and cosine can be used to solve problems involving any triangle.

Obtuse angles: An obtuse angle cannot occur in a right-angled triangle. So we must redefine the sine and cosine for such angles. In the diagram, OP makes an obtuse angle θ with the positive x-axis. PN is perpendicular to ON. Let PN = $+b$ and ON = $-a$.

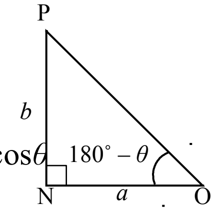


Then we define:

$$\sin\theta = \frac{NP}{OP} = \frac{+b}{OP} \qquad \cos\theta = \frac{ON}{OP} = \frac{-a}{OP}$$

Now $\angle NOP = 180˚ - \theta$. $\sin(180° - \theta) = \frac{NP}{OP} = \frac{b}{OP} = \sin\theta$

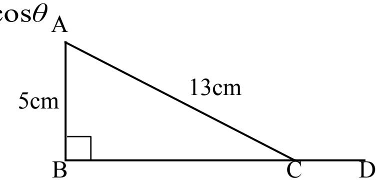
$$\cos(180° - \theta) = \frac{ON}{OP} = \frac{a}{OP} = -\cos\theta$$



Therefore if θ is obtuse,

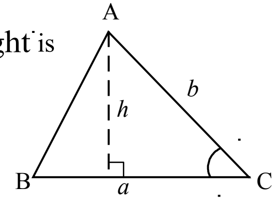
$$\boxed{\sin(180° - \theta) = \sin\theta \qquad \cos(180° - \theta) = -\cos\theta}$$

Example 5 In the diagram, ABC is a right-angled triangle in which AB = 5cm and AC = 13cm. BCD is a straight line, find $\sin\angle ACD$.



$$\sin\angle ACD = \sin(180° - \angle ACB) = \sin\angle ACB = \frac{AB}{AC} = \frac{5}{13}$$

Area of a triangle: The common formula $\frac{1}{2} \times \text{base} \times \text{height}$ is not always convenient as we may not know the height. We are developing this formula further so that we can calculate the area of a triangle given two sides and an included angle.



Suppose the known sides are a and b, and C is the angle between them.

Side a is taken as the base. The perpendicular height is given by $\sin C = \frac{h}{b}$.

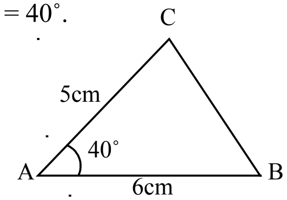
Therefore $h = b\sin C$. Then

$$\boxed{\text{Area of } \triangle ABC = \frac{1}{2}ab\sin C}$$

i.e. Area of a triangle = half × product of any two sides × sine of the included angle

The area of △ABC can be written as $\frac{1}{2}ab\sin C$, $\frac{1}{2}bc\sin A$ or $\frac{1}{2}ca\sin B$.

Example 6 In △ABC, AB = 6cm, AC = 5cm and ∠BAC = 40˚.
Find the area of the triangle.



$$\text{Area} = \frac{1}{2} AB \times AC \times \sin\angle BAC$$

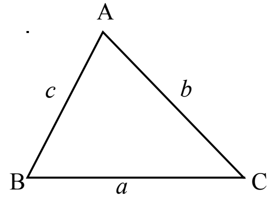
$$= \frac{1}{2} \times 6 \times 5 \times \sin 40°$$

$= 9.6418 = 9.64\text{ cm}^2$ (to 3sf)

Sine rule: $\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ or } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$

The area of △ABC can be written as $\frac{1}{2}ab\sin C$, $\frac{1}{2}bc\sin A$ or $\frac{1}{2}ca\sin B$.

It follows that $\frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$

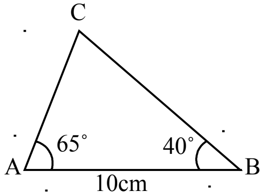


Dividing by $\frac{1}{2}abc$ gives $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

This can be rearranged as $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

The sine rule can be used to find a side when one side and any two angles are known, or to find an angle when two sides and an opposite angle are known.
The sine rule shows that the sides are proportional to the sines of the opposite angles, i.e. the shortest side is opposite the smallest angle, etc.

Example 7 In △ABC, AB = 10cm, ∠CAB = 65˚ and ∠ABC = 40˚. Calculate the lengths of BC and AC.



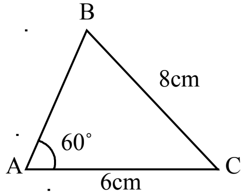
∠ACB = 180˚ – 65˚ – 40˚ = 75˚

Use the sine rule: $\frac{BC}{\sin 65°} = \frac{AC}{\sin 40°} = \frac{10}{\sin 75°}$

$$BC = \frac{10 \times \sin 65°}{\sin 75°} = 9.3827 = 9.38 \text{ cm (to 3sf)}$$

$$AC = \frac{10 \times \sin 40°}{\sin 75°} = 6.6546 = 6.65 \text{ cm (to 3sf)}$$

Example 8 In △ABC, AC = 6cm, BC = 8cm and ∠A = 60˚. Calculate the size of ∠B.



The sine rule gives $\frac{\sin A}{BC} = \frac{\sin B}{AC}$

$$\sin B = \frac{AC \times \sin A}{BC} = \frac{6 \times \sin 60°}{8} = 0.6495$$

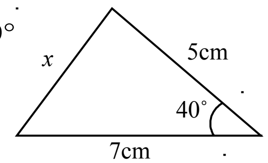
∴ ∠B = sin^{-1}0.6495 = 40.5˚

Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$

This formula can be rearranged as $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$.

This cosine rule can be used to find the third side when two sides and the included angle are known, or to find an angle when all the three sides are known.

Example 9 Find x in the diagram.

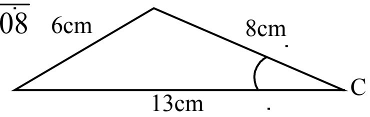


$$x^2 = 5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 40°$$
$$= 25 + 49 - 70 \times \cos 40°$$
$$= 74 - 53.62 = 20.38$$
$$\therefore x = \sqrt{20.38} = 4.5 \text{ cm}$$

Example 10 Find the smallest angle in the triangle which has sides of length 6cm, 8cm and 13cm.

The smallest angle in a triangle is opposite the shortest side.
In the given triangle, the smallest angle is opposite the 6cm side.
Let this angle be C. Then $c = 6$, and we can take $a = 8$ and $b = 13$.

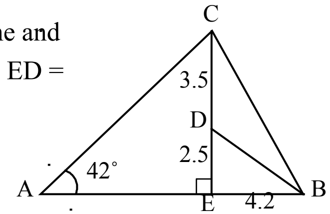
$$\cos C = \frac{8^2 + 13^2 - 6^2}{2 \times 8 \times 13} = \frac{64 + 169 - 36}{208} = \frac{197}{208}$$



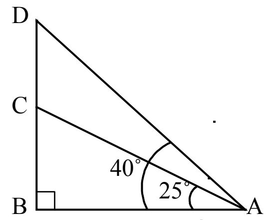
$$\therefore \hat{C} = \cos^{-1}\frac{197}{208} = 18.7°$$

Exercise 16

1 In the diagram on the right, CDE is a vertical line and AEB is a horizontal line. Given that EB = 4.2m, ED = 2.5m, DC = 3.5m and ∠CAE = 42˚, calculate:
(a) DB
(b) ∠ECB
(c) AC

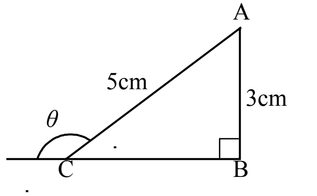


2 In the diagram on the right, BCD is a vertical line and AB is a horizontal line. The point A is 20cm away from B, the angle of elevation of C from A is 25˚ and the angle of elevation of D from A is 40˚. Find the length of CD, correct to 2 significant figures.

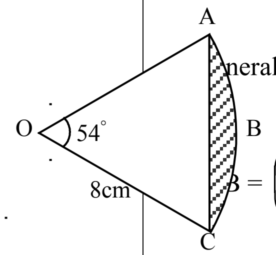


3 In the diagram below, write down the value of cosθ.

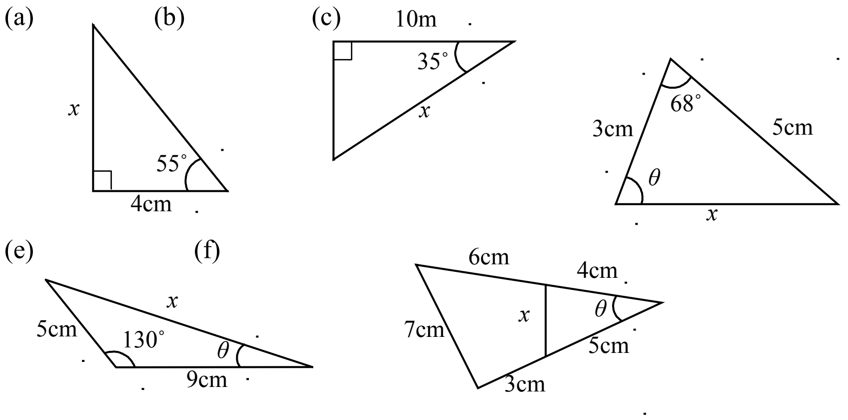
simsiky@gmail.com



4 In the diagram, ABC is a segment of a circle radius 8cm and centre O, where ∠AOC = 54°. Find the area of the segment.



5 In the diagrams, find the labelled sides and angles.



25 Matrices

A matrix is an array of numbers. The numbers of an array are called **elements**.

(ア) Calculations of Matrices

<u>Order of a matrix</u>: is given by "the number of **rows** by the number of **columns**".

e.g. $\begin{pmatrix} 1 & 2 \end{pmatrix}$ is a 1 × 2 matrix. $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is a 2 × 1 matrix.

$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ is a 2 × 2 matrix. $\begin{pmatrix} 1 & 2 & -2 & 10 \\ 5 & 1 & 9 & -4 \\ 3 & 4 & 7 & -7 \end{pmatrix}$ is a 3 × 4 matrix.

<u>Addition and subtraction of matrices</u>: Matrices of the same order can be added or subtracted by adding or subtracting corresponding elements.

In general, 2 × 2 matrix A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and 2 × 2 matrix B= $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

Example 1 Simplify: (a) $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ 0 & 4 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 6 & -4 \\ -3 & 2 \\ 2 & 0 \end{pmatrix}$

(a) $\begin{pmatrix} 1+2 & 4+3 \\ 2+(-3) & 3+1 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ -1 & 4 \end{pmatrix}$

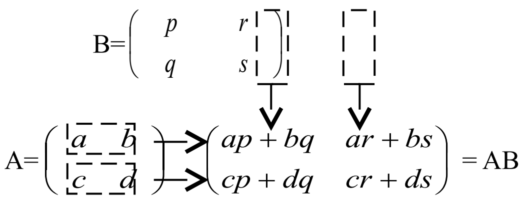
(b) $\begin{pmatrix} 1+3-6 & 4+(-2)-(-4) \\ 2+0-(-3) & 5+4-2 \\ 3+(-1)-2 & 6+2-0 \end{pmatrix} = \begin{pmatrix} -2 & 6 \\ 5 & 7 \\ 0 & 8 \end{pmatrix}$

<u>Scalar multiplication of a matrix</u>: (A scalar is a number written in front of a matrix.) Each element of that matrix is multiplied by that number.

e.g. $3\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 3\times 2 & 3\times 0 \\ 3\times 3 & 3\times(-1) \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 9 & -3 \end{pmatrix}$

Multiplication of matrices: Matrices can be multiplied if the number of columns in the first matrix is equal to the number of rows in the second matrix. It is done by laying each row of the first matrix against each column of the second matrix, multiplying the pairs of elements and adding the results together to make a single matrix. The order of the product of two matrices can be obtained from the number of rows in the first matrix by the number of columns in the second matrix.

In general, 2 × 2 matrix A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and 2 × 2 matrix B= $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$



Example 2 Find: (a) $\begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ -5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 2 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

(a) $\begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \times \begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 1\times 4 + 3\times(-5) \\ -2\times 4 + 2\times(-5) \end{pmatrix} = \begin{pmatrix} -11 \\ -18 \end{pmatrix}$

2 × 2 2 × 1 order of product (2 × 1)

(b) $\begin{pmatrix} 1 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1\times 3 + 2\times 2 & 1\times 0 + 2\times 4 \end{pmatrix} = \begin{pmatrix} 7 & 8 \end{pmatrix}$

1 × 2 2 × 2 order of product (1 × 2)

(c) $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1\times 0 + 3\times 1 & 1\times(-1) + 3\times 2 \\ 2\times 0 + 4\times 1 & 2\times(-1) + 4\times 2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$

2 × 2 2 × 2 order of product (2 ラ 2)

Example 3 If $\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} -1 & x \\ y & -2 \end{pmatrix} = \begin{pmatrix} 4 & -12 \end{pmatrix}$, find x and y.

Multiply $\begin{pmatrix} -2+3y & 2x-6 \end{pmatrix} = \begin{pmatrix} 4 & -12 \end{pmatrix}$

So $-2 + 3y = 4$ and $2x - 6 = -12$ $\therefore x = -3$ and $y = 2$

(イ) Inverse Matrix

Identity matrix: is a square matrix whose elements in the main diagonal (from the top left corner to the bottom right corner) are all '1' and the others are all '0' and is denoted by "*I*".

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2 × 2 identity matrix. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the 3 × 3 identity matrix.

For any square matrix A, $\mathbf{A}I = I\mathbf{A} = \mathbf{A}$.

Determinant of a matrix: is the product of elements in the main diagonal minus the product of the elements in the other diagonal (from the top right corner to the bottom left corner). The determinant of a matrix A is denoted by "det A".

In general, if A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\boxed{\det A = ad - bc}$

e.g. If A = $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$, det A = 2 × 4 – 3 × 1 = 8 – 3 = 5

A matrix whose determinant is zero is called a **singular** matrix.

Inverse of a matrix: is another matrix such that when the two matrices are multiplied together in any order, the result is the identity matrix. A^{-1} is the inverse of a matrix A if $\mathbf{A^{-1}A = AA^{-1}} = I$.

In general, if A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\boxed{A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

simsiky@gmail.com

e.g. If $A=\begin{pmatrix}2&3\\1&4\end{pmatrix}$, $A^{-1}=\frac{1}{5}\begin{pmatrix}4&-3\\-1&2\end{pmatrix}=\begin{pmatrix}\frac{4}{5}&-\frac{3}{5}\\-\frac{1}{5}&\frac{2}{5}\end{pmatrix}$

A singular matrix has no inverse because the determinant is zero.

e.g. The determinant of $\begin{pmatrix}2&4\\1&2\end{pmatrix}$ is $2 \times 2 - 4 \times 1 = 0$. So the inverse cannot be found.

Matrix method of solving simultaneous equations: Matrices can be used to solve simultaneous linear equations.

Example 4 Solve $2x - y = 5$

$x - 2y = 4$

Write the equations in matrix form. $\begin{pmatrix}2&-1\\1&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}5\\4\end{pmatrix}$

Let $A=\begin{pmatrix}2&-1\\1&-2\end{pmatrix}$.

Then $\det A = 2 \times (-2) - (-1) \times 1 = -4 + 1 = -3$. So $A^{-1}=\frac{1}{-3}\begin{pmatrix}-2&1\\-1&2\end{pmatrix}$.

Pre-multiply both sides by A^{-1}. (i.e. A^{-1} goes on the left.)

$$\frac{1}{-3}\begin{pmatrix}-2&1\\-1&2\end{pmatrix}\begin{pmatrix}2&-1\\1&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{-3}\begin{pmatrix}-2&1\\-1&2\end{pmatrix}\begin{pmatrix}5\\4\end{pmatrix}$$

Here $A^{-1}A = I$.

$$I\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{-3}\begin{pmatrix}-10+4\\-5+8\end{pmatrix}=\frac{1}{-3}\begin{pmatrix}-6\\3\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix}$$

$\therefore x = 2$ and $y = -1$

Exercise 17

1 Simplify: (a) $\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}1\\-2\end{pmatrix}$ (b) $\begin{pmatrix}5&2&-4\end{pmatrix}-\begin{pmatrix}2&-3&1\end{pmatrix}$ (c) $2\begin{pmatrix}0&-3\\2&2\end{pmatrix}+3\begin{pmatrix}1&2\\-1&3\end{pmatrix}$

(d) $3\begin{pmatrix}2&-1\end{pmatrix}-2\begin{pmatrix}-3&4\end{pmatrix}$ (e) $4\begin{pmatrix}0&2&4\\1&3&5\end{pmatrix}-2\begin{pmatrix}3&2&0\\-2&1&4\end{pmatrix}$

2 Express as single matrices: (a) $\begin{pmatrix}2&1\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}$ (b) $\begin{pmatrix}3\\2\end{pmatrix}\begin{pmatrix}2&1\end{pmatrix}$ (c) $\begin{pmatrix}2&1\end{pmatrix}\begin{pmatrix}-2&2\\1&-3\end{pmatrix}$

(d) $\begin{pmatrix}1&3\\2&2\end{pmatrix}\begin{pmatrix}2&3\\-1&1\end{pmatrix}$ (e) $\begin{pmatrix}3&1\\-1&2\end{pmatrix}\begin{pmatrix}2&-3&4\\2&1&-2\end{pmatrix}$

3 Given that $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $B=\begin{pmatrix}-1&-2\\2&2\end{pmatrix}$, calculate: (a) A + B (b) A – 2B

(c) B + 3A (d) AB (e) BA (f)$A^2 - B^2$

4 $A=\begin{pmatrix}1&2\\a&-1\end{pmatrix}$ and $B=\begin{pmatrix}2&1\\1&b\end{pmatrix}$. If AB = BA, find the values of a and b.

5 If the determinant of a matrix $\begin{pmatrix}2a&-4\\a&1\end{pmatrix}$ is 12, find the value of a.

6 If a matrix $\begin{pmatrix}x&3\\x+2&x+8\end{pmatrix}$ does not have the inverse, find the value of x.

7 Using matrices, solve the simultaneous equations.

(a) $x - y = 1$ (b) $2x + y = 7$ (c) $2x - 3y = 14$

$2x + 5y = 16$ $3x - 2y = 7$ $3x + 2y = -5$

26 Vectors

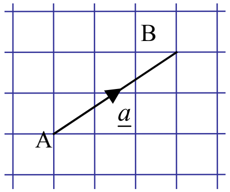
(ア) Basic Vector

Scalar and vector quantities: There are two types of quantities.

A **scalar** is a quantity that has magnitude only.

A **vector** is a quantity that has both magnitude and direction.

Notation: A vector can be drawn as a **directed line**. The length of a line represents the magnitude of the vector and the arrow indicates its direction.



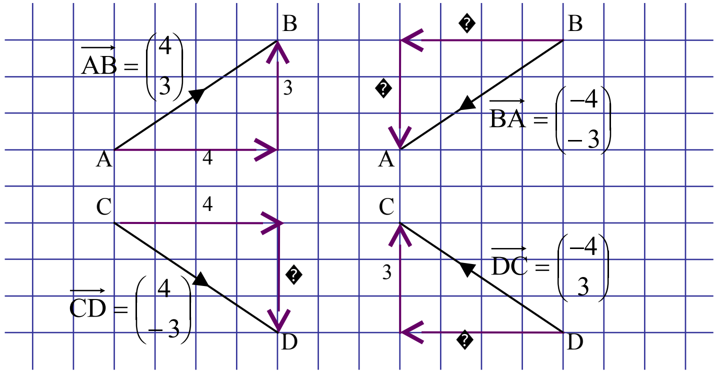
This vector can be written as $\overrightarrow{AB}$.

It can also be written as $\underline{a}$.

A vector can be written as a **column vector**, in the form $\begin{pmatrix} x \\ y \end{pmatrix}$.

The top number (x) in a column vector represents movement parallel to the x-axis and the bottom number (y) represents movement parallel to the y-axis.

Movement to the right and up is positive, and movement to the left and down is negative.

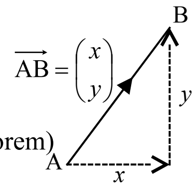


$\overrightarrow{AB} = -\overrightarrow{BA}$ $\overrightarrow{CD} = -\overrightarrow{DC}$

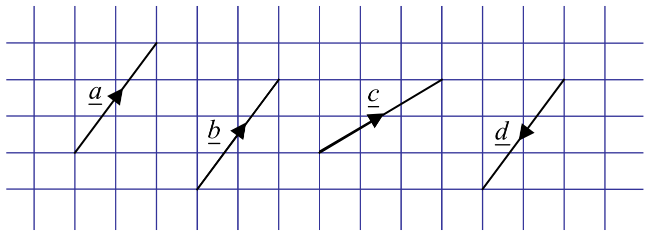
Magnitude of a vector: is the length of the vector.

The magnitude of vector $\overrightarrow{AB}$ is written as $|\overrightarrow{AB}|$.

In general, if $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$, $|\overrightarrow{AB}| = \sqrt{x^2 + y^2}$ (∵Pythagoras theorem)



Equal vectors: are vectors whose magnitudes and directions are the same. As vectors are usually independent of position, they can start at any point.



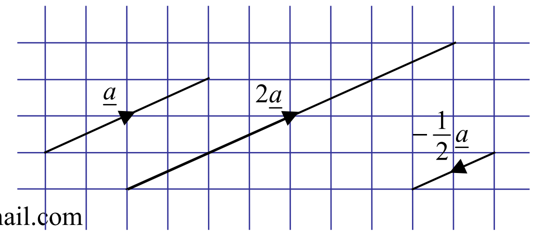
$\underline{a} = \underline{b}$

$\underline{a} \neq \underline{c}$

$\underline{a} \neq \underline{d}$ $(\underline{a} = -\underline{d})$

Scalar multiplication: is done by expressing a vector as a column vector and multiplying each number by the scalar.

e.g. $2\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \times 4 \\ 2 \times 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$

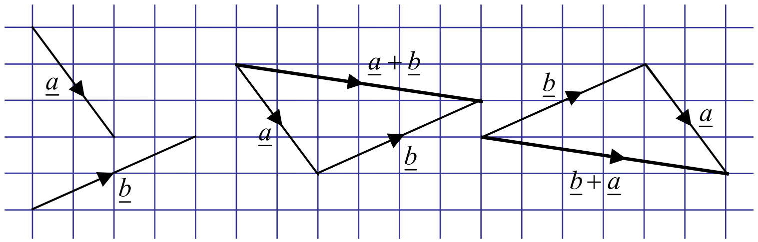


simsiky@gmail.com

$$-\frac{1}{2}\begin{pmatrix}4\\2\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\times 4\\-\frac{1}{2}\times 2\end{pmatrix}=\begin{pmatrix}-2\\-1\end{pmatrix}$$

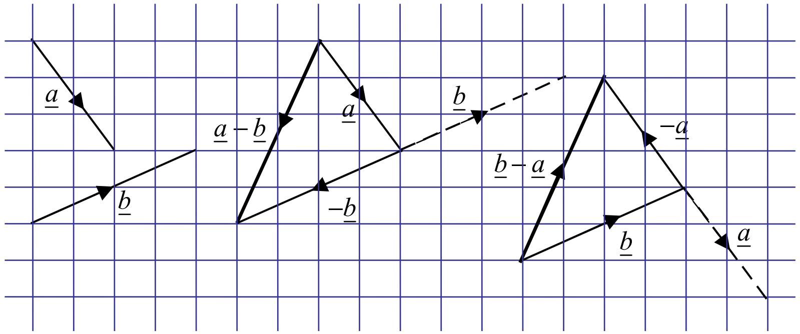
Addition and subtraction of vectors: Vectors can be added or subtracted.

To add vectors $\underline{a}$ and $\underline{b}$, draw the second vector ($\underline{b}$) at the end of the first vector ($\underline{a}$) and draw a new vector from the beginning of $\underline{a}$ to the end of $\underline{b}$.



$$\underline{a}+\underline{b}=\underline{b}+\underline{a}$$

To subtract vectors ($\underline{a}$ – $\underline{b}$), let $\underline{a}$ + (–$\underline{b}$). So the addition above can be applied.



$$\underline{a}-\underline{b}=-(\underline{b}-\underline{a})$$

To add or subtract column vectors, add or subtract the top number in each vector, then add or subtract the bottom number in each vector.

e.g. $\underline{a}=\begin{pmatrix}2\\-3\end{pmatrix}$ $\underline{b}=\begin{pmatrix}4\\2\end{pmatrix}$

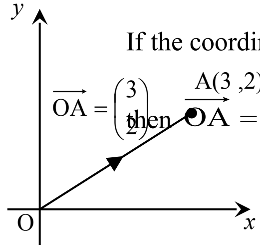
$$\underline{a}+\underline{b}=\begin{pmatrix}2\\-3\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}=\begin{pmatrix}2+4\\-3+2\end{pmatrix}=\begin{pmatrix}6\\-1\end{pmatrix}\qquad \underline{b}+\underline{a}=\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}4+2\\2+-3\end{pmatrix}=\begin{pmatrix}6\\-1\end{pmatrix}$$

$$\underline{a}-\underline{b}=\begin{pmatrix}2\\-3\end{pmatrix}-\begin{pmatrix}4\\2\end{pmatrix}=\begin{pmatrix}2-4\\-3-2\end{pmatrix}=\begin{pmatrix}-2\\-5\end{pmatrix}\qquad \underline{b}-\underline{a}=\begin{pmatrix}4\\2\end{pmatrix}-\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}4-2\\2-(-3)\end{pmatrix}=\begin{pmatrix}2\\5\end{pmatrix}$$

(イ) Vector Geometry

Position vectors: are vectors which start at a known point (usually origin), and its finishing point gives a position relative to that starting point.

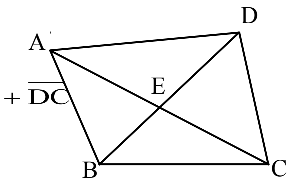
e.g.



If the coordinates of A are (3, 2),

then $\overrightarrow{OA}=\begin{pmatrix}3\\2\end{pmatrix}$

Vector geometry: Vectors can be used to solve geometrical problems.



Example 1 In the figure, find directed line equal to the following: (a) $\overrightarrow{AE}+\overrightarrow{EC}$ (b) $\overrightarrow{BD}+\overrightarrow{DE}$ (c) $\overrightarrow{AB}+\overrightarrow{BD}+\overrightarrow{DC}$

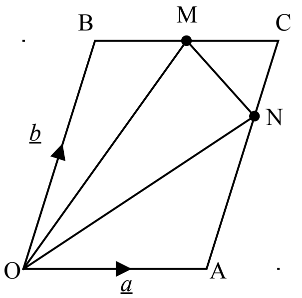
(a) $\overrightarrow{AE}+\overrightarrow{EC}=\overrightarrow{AC}$ (b) $\overrightarrow{BD}+\overrightarrow{DE}=\overrightarrow{BE}$

(c) $\overrightarrow{AB}+\overrightarrow{BD}+\overrightarrow{DC}=\overrightarrow{AD}+\overrightarrow{DC}=\overrightarrow{AC}$

Example 2 The coordinates of P are (3, –1) and the coordinates of Q are (1, 3). Find the vector $\overrightarrow{PQ}$.

$$\overrightarrow{PQ}=\overrightarrow{PO}+\overrightarrow{OQ}=-\overrightarrow{OP}+\overrightarrow{OQ}=-\begin{pmatrix}3\\-1\end{pmatrix}+\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}-2\\4\end{pmatrix}$$

Example 3 OACB is a parallelogram in which $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. M is the mid-point of BC and N is a point on AC such that AN : NC = 2 : 1. Express the following in terms of $\underset{\sim}{a}$ and / or $\underset{\sim}{b}$.



(a) $\overrightarrow{BC}$ (b) $\overrightarrow{BM}$ (c) $\overrightarrow{OM}$ (d) $\overrightarrow{ON}$ (e) $\overrightarrow{MN}$

(a) $\overrightarrow{BC} = \overrightarrow{OA} = \underset{\sim}{a}$

(b) $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\underset{\sim}{a}$

(c) $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM} = \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}$

(d) $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AC} = \underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b}$

(e) $\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON} = -\overrightarrow{OM} + \overrightarrow{ON} = -\left(\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}\right) + \underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b}$

$= \underset{\sim}{a} - \frac{1}{2}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b} - \underset{\sim}{b} = \frac{1}{2}\underset{\sim}{a} - \frac{1}{3}\underset{\sim}{b}$

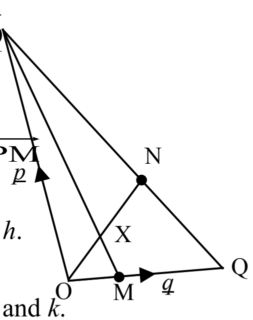
Example 4 In the diagram, $\overrightarrow{OP} = \underset{\sim}{p}$, $\overrightarrow{OQ} = \underset{\sim}{q}$, $\overrightarrow{OM} = \frac{1}{3}\overrightarrow{OQ}$ and PN : NQ = 3 : 2.

(a) Express in terms of $\underset{\sim}{p}$ and / or $\underset{\sim}{q}$. (i) $\overrightarrow{PQ}$ (ii) $\overrightarrow{ON}$ (iii) $\overrightarrow{PM}$

(b) Given that $\overrightarrow{OX} = h\overrightarrow{ON}$, express $\overrightarrow{OX}$ in terms of $\underset{\sim}{p}$, $\underset{\sim}{q}$ and h.

(c) Given also that $\overrightarrow{PX} = k\overrightarrow{PM}$, express $\overrightarrow{OX}$ in terms of $\underset{\sim}{p}$, $\underset{\sim}{q}$ and k.

(d) Find the values of h and k.



(a)

(i) $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\overrightarrow{OP} + \overrightarrow{OQ} = -\underset{\sim}{p} + \underset{\sim}{q}$

(ii) $\overrightarrow{ON} = \overrightarrow{OP} + \overrightarrow{PN} = \overrightarrow{OP} + \frac{3}{5}\overrightarrow{PQ} = \underset{\sim}{p} + \frac{3}{5}\left(-\underset{\sim}{p} + \underset{\sim}{q}\right) = \frac{2}{5}\underset{\sim}{p} + \frac{3}{5}\underset{\sim}{q}$

(iii) $\overrightarrow{PM} = \overrightarrow{PO} + \overrightarrow{OM} = -\overrightarrow{OP} + \frac{1}{3}\overrightarrow{OQ} = -\underset{\sim}{p} + \frac{1}{3}\underset{\sim}{q}$

(b) $\overrightarrow{OX} = h\overrightarrow{ON} = h\left(\frac{2}{5}\underset{\sim}{p} + \frac{3}{5}\underset{\sim}{q}\right)$

(c) $\overrightarrow{OX} = \overrightarrow{OP} + \overrightarrow{OX} = \overrightarrow{OP} + k\overrightarrow{PM} = \underset{\sim}{p} + k\left(-\underset{\sim}{p} + \frac{1}{3}\underset{\sim}{q}\right) = (1-k)\underset{\sim}{p} + \frac{k}{3}\underset{\sim}{q}$

(d) From (b) and (c), $h\left(\frac{2}{5}\underset{\sim}{p} + \frac{3}{5}\underset{\sim}{q}\right) = (1-k)\underset{\sim}{p} + \frac{k}{3}\underset{\sim}{q}$.

Compare the coefficients of $\underset{\sim}{p}$ and $\underset{\sim}{q}$.

$\frac{2}{5}h = 1 - k$ ------------(i)

$\frac{3}{5}h = \frac{k}{3}$ ----------------(ii)

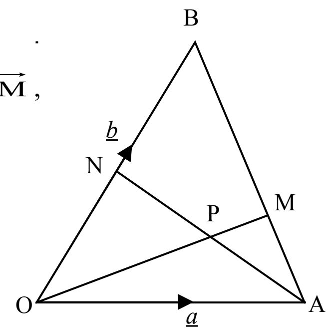
(ii) → $k = \frac{9}{5}h$ Substitute this for k into (i) $\frac{2}{5}h = 1 - \frac{9}{5}h$ $\therefore h = \frac{5}{11}, k = \frac{9}{11}$

Exercise 18

1 If $\underset{\sim}{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, find: (a) $\underset{\sim}{a} + \underset{\sim}{b}$ (b) $\underset{\sim}{a} - \underset{\sim}{b}$ (c) $3\underset{\sim}{a} + 2\underset{\sim}{b}$ (d) $|\underset{\sim}{a}|$ (e) $|\underset{\sim}{b}|$ (f) $|\underset{\sim}{a} + \underset{\sim}{b}|$ (g) $|\underset{\sim}{a} - \underset{\sim}{b}|$

2 In △OAB, $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

(a) Given that M is a point on AB such that $\overrightarrow{AB} = 3\overrightarrow{AM}$, express in terms of $\underset{\sim}{a}$ and / or $\underset{\sim}{b}$. (i) $\overrightarrow{AB}$ (ii) $\overrightarrow{OM}$

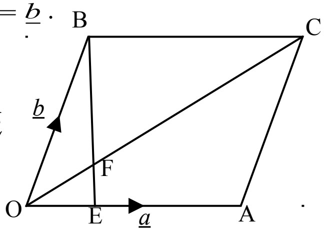


simsiky@gmail.com

(b) Given that N is the midpoint of OB, express $\overrightarrow{AN}$ in terms of $\underline{a}$ and $\underline{b}$.

(c) Given that $\overrightarrow{OM}$ meets $\overrightarrow{AN}$ at P and $\overrightarrow{AP} = h\overrightarrow{AN}$, express $\overrightarrow{OP}$ in terms of $\underline{a}$, $\underline{b}$ and h.

3 OACB is a parallelogram in which $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$. The point E on OA is such that OE : EA = 1 : 2.



(a) Express in terms of $\underline{a}$ and / or $\underline{b}$. (i) $\overrightarrow{OE}$ (ii) $\overrightarrow{BE}$

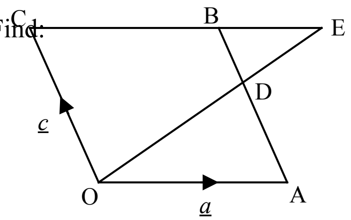
(b) OC and BE meet at F. Given that $\overrightarrow{BF} = k\overrightarrow{BE}$, express in terms of $\underline{a}$, $\underline{b}$ and k. (i) $\overrightarrow{BF}$ (ii) $\overrightarrow{OF}$

(c) Given also that $\overrightarrow{OF} = h\overrightarrow{OC}$, express $\overrightarrow{OF}$ in terms of $\underline{a}$, $\underline{b}$ and h.

(d) Find the values of h and k.

(e) Find the ratio of OF : OC

4 OABC is parallelogram in which $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OC} = \underline{c}$. The point D is on AB such that AD : DB = 2 : 1. When the line OD is produced, it meets the line CB at E such that $\overrightarrow{DE} = h\overrightarrow{OD}$ and $\overrightarrow{BE} = k\overrightarrow{CB}$. Find:



(a) $\overrightarrow{BE}$ in terms of $\underline{a}$ and k

(b) $\overrightarrow{DE}$ in terms of $\underline{a}$, $\underline{c}$ and h

(c) the values of h and k

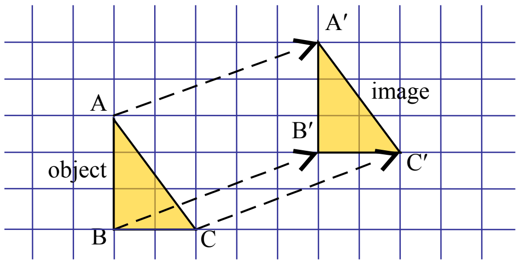
27 Transformation

A transformation is a geometrical operation which maps a set of points (object) onto another set (image). Transformations are described matrices.

(ア) Translation

A translation is a transformation which moves every point of the object in a straight line to another position. The translated image is the same size and shape as the object. A translation is usually denoted by T.

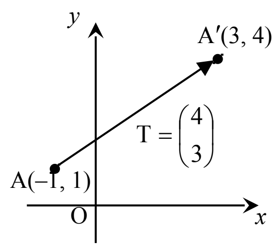
A translation can be described by a column vector: $T = \begin{pmatrix} x \\ y \end{pmatrix}$.



Example 1 What is the image A' of the point A(–1, 1) under the translation $T = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$?

The position vector of A is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Then the position vector of A′ is given by $\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.



So the coordinates of A' are (3, 4)

Example 2 If the point (3, –2) is translated to (5, –5), what is the translation?

$$\begin{pmatrix}3\\-2\end{pmatrix}+T=\begin{pmatrix}5\\-5\end{pmatrix} \quad \therefore T=\begin{pmatrix}5\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}2\\-3\end{pmatrix}$$

(イ) Transformations Represented by 2 × 2 Matrices

A translation is the only transformation described by a column vector. All the other transformations are described by 2 × 2 matrices.

The image A′(x', y') of a point A(x, y) under a transformation P is found by pre-multiplying its position vector $\begin{pmatrix}x\\y\end{pmatrix}$ by the matrix $P=\begin{pmatrix}a & b\\c & d\end{pmatrix}$.

$$A' = PA \qquad \begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}ax+by\\cx+dy\end{pmatrix} \qquad \therefore (x', y') = (ax+by, cx+dy)$$

Example 3 Find the 2 × 2 matrix which maps the points (1, 1) and (0, –2) onto the points (3, –1) and (–2, 4) respectively.

Let the matrix be $\begin{pmatrix}a & b\\c & d\end{pmatrix}$. Then $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3\\-1\end{pmatrix}$ and

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}0\\-2\end{pmatrix}=\begin{pmatrix}-2\\4\end{pmatrix}$$

These give two sets of simultaneous equations. $\begin{pmatrix}a+b\\c+d\end{pmatrix}=\begin{pmatrix}3\\-1\end{pmatrix}$ and

$$\begin{pmatrix}-2b\\-2d\end{pmatrix}=\begin{pmatrix}-2\\4\end{pmatrix}$$

Solving these: $a = 2$, $b = 1$, $c = 1$ and $d = -2$. Therefore $\begin{pmatrix}2 & 1\\1 & -2\end{pmatrix}$

The matrix of a transformation can be found if the images of the points (1, 0) and (0, 1) are known. Because if the matrix is $\begin{pmatrix}a & b\\c & d\end{pmatrix}$, then $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}a\\c\end{pmatrix}$ which is the first column of the matrix, and $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}b\\d\end{pmatrix}$ which is the second column of the matrix.

> If (1, 0) is mapped onto (a, c) and (0, 1) onto (b, d), the matrix is $\begin{pmatrix}a & b\\c & d\end{pmatrix}$.

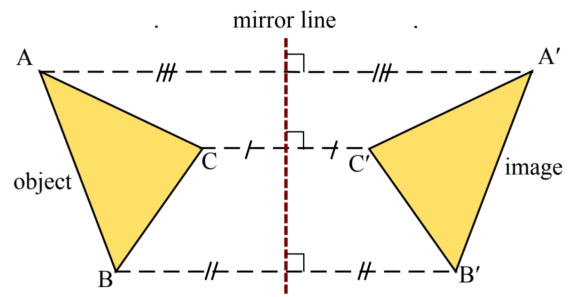
Example 4 A transformation maps (1, 0) onto (–1, 3) and (0, 1) onto (–2, 1). What is the matrix of this transformation?

The first column of the matrix is $\begin{pmatrix}-1\\3\end{pmatrix}$ and the second column is $\begin{pmatrix}-2\\1\end{pmatrix}$.

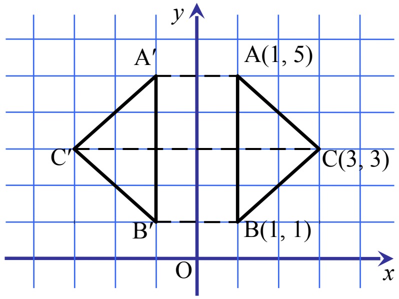
Therefore the matrix is $\begin{pmatrix}-1 & -2\\3 & 1\end{pmatrix}$.

(ウ) Reflection

A reflection is a transformation in which any two corresponding points in the object and the image are the same distance from, and at right angles to, a straight line which is called **mirror line**.



Example 5 △ABC has vertices A(1, 5), B(1, 1) and C(3, 3). Find the coordinates of the images of A, B and C under reflection in the y-axis.



The coordinates of the images are

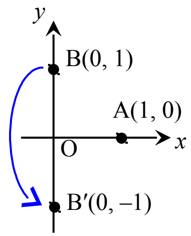
A′(–1, 5)

B′(–1, 1)

C′(–3, 3)

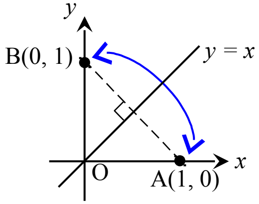
Example 6 Find the matrices for reflections in the following:
(a) the x-axis (b) the line $y = x$ (c) the line $y = -x$

Find the images of A(1, 0) and B(0, 1).



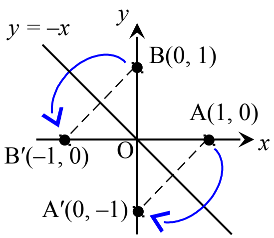
(a) The image of A is invariant as it lies on the x-axis.
The point B is mapped onto B′(0, –1).

Therefore the matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.



(b) The point A is mapped onto B(0, 1).
The point B is mapped onto A(1, 0)

Therefore the matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

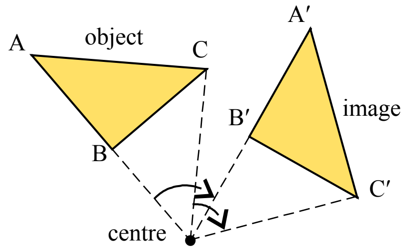


(c) The point A is mapped onto A′(0, –1).
The point B is mapped onto B′(–1, 0)

. Therefore the matrix is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

(エ) Rotation

A rotation is a transformation in which an object is turned around a fixed point which is called the **centre of rotation**.

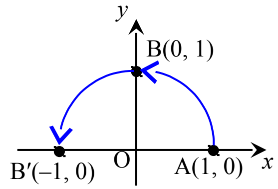


The angle through which an object is turned is called the angle of rotation. The direction of the rotation, clockwise or anticlockwise, must be stated. An anticlockwise rotation is usually said to be positive.

Only if the centre of rotation is the origin, the matrix for rotation can be found.

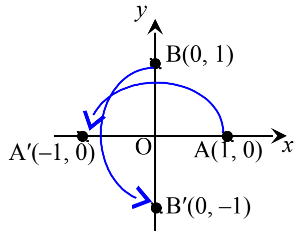
Example 7 Find the matrices for rotations about O(0, 0) of the following:
(a) 90˚ anticlockwise (b) 180˚ anticlockwise (c) 270˚ anticlockwise

Find the images of A(1, 0) and B(0, 1).



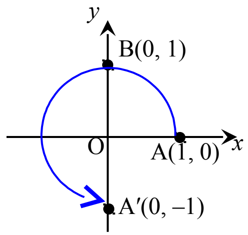
(a) The point A is mapped onto B(0 ,1).
The point B is mapped onto B′(–1, 0).

Therefore the matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.



(b) The point A is mapped onto A′(–1, 0).
The point B is mapped onto B′(0, –1)

Therefore the matrix is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

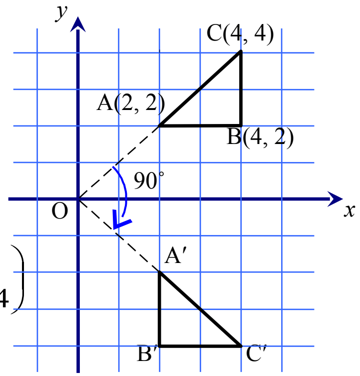


(c) The point A is mapped onto A′(0, –1).
The point B is mapped onto A(1, 0)

. Therefore the matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

This is equal to a rotation of 90˚ clockwise.

Example 8 △ABC has vertices A(2, 2), B(4, 2) and C(4, 4). Find the image of △ABC under a rotation about O through 90˚ clockwise.



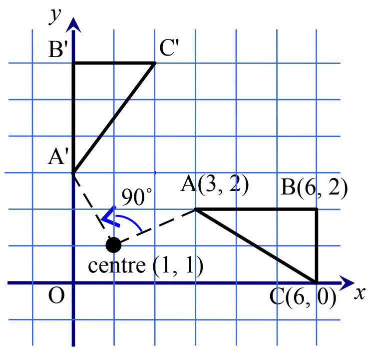
From Example 7, the matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Multiply the three column vectors for A, B and C as 2 × 3 matrix:

$$\begin{matrix} & \text{A} & \text{B} & \text{C} & & \text{A}' & \text{B}' & \text{C}' \end{matrix}$$
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 4 & 4 \\ 2 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 4 \\ -2 & -4 & -4 \end{pmatrix}$$

∴A′(2, –2), B′(2, –4) and C′(4, –4)

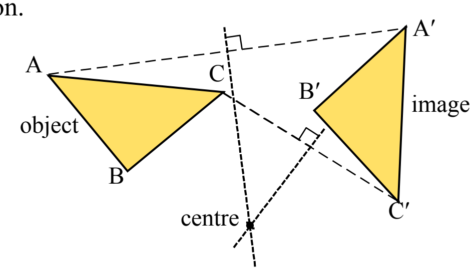
Example 9 △ABC has vertices A(3, 2), B(6, 2) and C(6, 0). Draw the image of △ABC under an anticlockwise rotation of 90˚ centre (1, 1).



The centre of rotation is not the origin.
So the matrix of this rotation cannot be found.

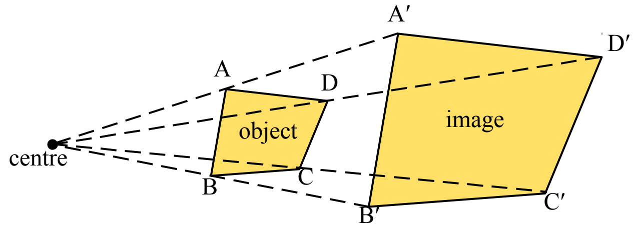
First join A to the centre (1, 1).
Then with angle of 90˚ and the same distance between A and (1, 1), draw A′.
Do the same for B and C as shown in the diagram on the right.

To find the centre of rotation: Join any corresponding two points in the object and the image and construct the perpendicular bisector of each line. The point of intersection is the centre of rotation.



(オ) Enlargement

An enlargement is a transformation which changes the size but not the shape of an object.



When corresponding points in the object and the image are joined by straight lines, all those lines meet at a point which is called the **centre of enlargement**. The centre of enlargement can be inside, on the edge of, or outside the object.

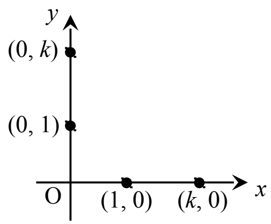
The amount by which an object is enlarged is called the **scale factor** $\left(= \dfrac{\text{image length}}{\text{object length}}\right)$.

Only if the centre of enlargement is the origin, the matrix for enlargement can be found.

Example 10 Find the matrix for an enlargement centre O(0, 0) and scale factor k.

The point (1, 0) is mapped onto (k, 0).
The point (0, 1) is mapped onto (0, k)

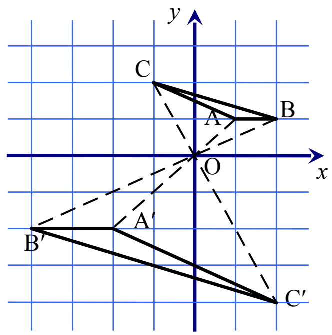
Therefore the matrix is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.



Example 11 △ABC has vertices A(1, 1), B(2, 1) and C(–1, 2). Find the image of △ABC under an enlargement with centre O(0, 0) and scale factor –2.

From Example 10, the matrix is $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$.

Multiply the three column vectors for A, B and C as 2 × 3 matrix:



simsiky@gmail.com

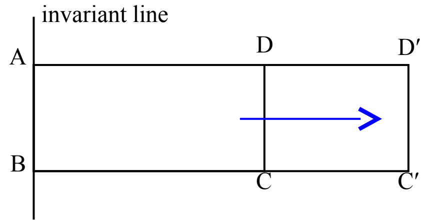
$$\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}\overset{A}{1} & \overset{B}{2} & \overset{C}{-1}\\ 1 & 1 & 2\end{pmatrix}=\begin{pmatrix}\overset{A'}{-2} & \overset{B'}{-4} & \overset{C'}{2}\\ -2 & -2 & -4\end{pmatrix}$$

∴A′(–2, –2), B′(–4, –2) and C′(2, –4)

When the scale factor is negative, the object and its image are on opposite sides of the centre of enlargement.

(カ) Stretch

A stretch is an enlargement in one direction from a given line. The given line is the invariant line or axis.



Example 12 The matrix S is a stretch such that the y-axis is invariant and the point (1, 2) is mapped onto (2, 2). (a) Find the matrix S. (b) If △PQR has vertices P(2, 1), Q(3, 5) and R(5, 1). Find the image of △PQR under the stretch S.

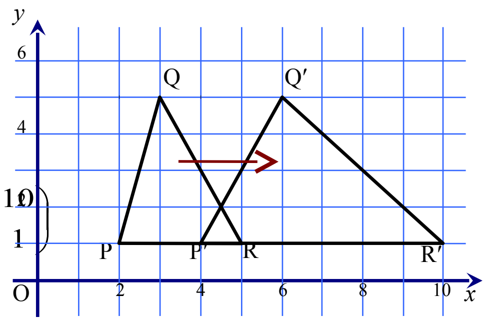
(a) Take A as (1, 0) and B as (0, 1). Now OB is invariant.
If the point(1, 2) moves 1unit to (2, 2), A(1, 0) also moves 1unit to A′(2, 0).

Therefore the matrix is $\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$.

(b) Multiply the three column vectors for P, Q and R as 2 × 3 matrix:

$$\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\overset{P}{2} & \overset{Q}{3} & \overset{R}{5}\\ 1 & 5 & 1\end{pmatrix}=\begin{pmatrix}\overset{P'}{4} & \overset{Q'}{6} & \overset{R'}{10}\\ 1 & 5 & 1\end{pmatrix}$$

∴P′(4, 1), Q′(6, 5) and R′(10, 1)



In general,

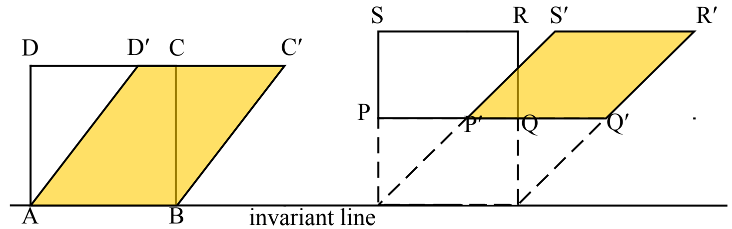
The matrix $\begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix}$ represents a stretch with x-axis as the invariant and scale factor k.

The matrix $\begin{pmatrix}k & 0\\ 0 & 1\end{pmatrix}$ represents a stretch with y-axis as the invariant and scale factor k.

$$\text{Scale factor} = \frac{\text{distance of the image of a point from the invariant line}}{\text{distance of the point from the invariant line}}$$

(キ) Shear

A shear is a transformation which keeps a fixed line (invariant line), moves all other points parallel to the invariant line and maps every straight line onto a straight line.



In general,

the matrix $\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}$ represents a shear with the x-axis as the invariant and shear factor k.

the matrix $\begin{pmatrix}1 & 0\\ k & 1\end{pmatrix}$ represents a shear with the y axis as the invariant and shear factor k.

$$\text{Shear factor} = \frac{\text{distance a point moves to its image}}{\text{distance of the point from the invariant line}}$$

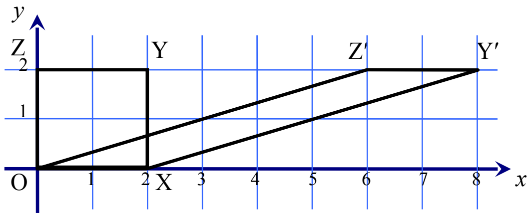
The shear factor indicates that the distance a point moves to its image is proportional to its distance from the invariant line.

Example 13 A rectangle OXYZ with vertices O(0, 0), X(2, 0), Y(2, 2) and Z(0, 2) is mapped onto the parallelogram OXY′Z′ with Y′(8, 2) and Z′(6, 2).

(a) Find the matrix of the shear.

(b) Describe the shear fully.

First draw the diagram to show the information above.



Take A as (1, 0) and B as (0, 1). Now OA is invariant. (∵the x-axis is invariant)

Now Y(2, 2) moves 6units, so B(0, 1) moves 3units to B′(0, 3). (∵proportion)

Therefore the matrix is $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

(b) Shear with x-axis as the invariant line and shear factor 3.

Exercise 19

1 A triangle ABC with vertices A(2, 2), B(4, 2) and C(3, 4) is mapped onto the triangle with vertices A′(–2, –1), B′(0, –1) and C′(–1, 1).

(a) Draw and label the triangle and its image.

(b) Describe the transformation which maps ABC onto A′B′C′.

2 A rectangle ABCD with vertices A(1, 1), B(4, 1), C(4, 3) and D(1, 3) is mapped onto A′B′C′D′ with vertices A′(1, –1), B′(4, –1), C′(4, –3) and D′(1, –3).

(a) Draw and label the triangle and its image.

(b) Describe the transformation which maps ABCD onto A′B′C′D′.

3 △ABC with vertices A(1, –1), B(3, –2) and C(3, –1) is mapped onto △A′B′C′ with vertices A′(6, 0), B′(7, 2) and C′(6, 2) by a rotation.

(a) Draw and label △ABC and △A′B′C′.

(b) Find the centre of rotation by construction.

(c) Find the angle of rotation and its direction.

4 △ABC with vertices A(1, 1), B(2, 1) and C(2, 2) is mapped onto △A′B′C′ with vertices A′(1, –3), B′(2, –3) and C′(2, –6) by a stretch. Find the matrix and the scale factor of the stretch.

5 (a) Draw and label △ABC whose vertices A(2, 2), B(4, 2) and C(2, 5).

(b) A transformation maps △ABC onto △DEF whose vertices D(5, 2), E(1, 2) and F(5, –4). Describe this transformation fully.

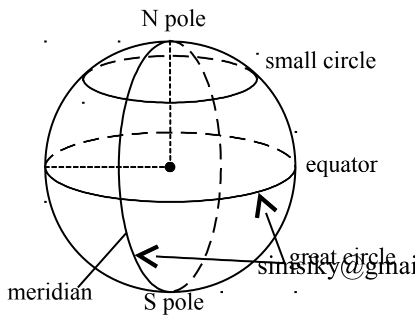
(c) A transformation maps △ABC onto △GHI whose vertices G(6, 2), H(8, 2) and I(12, 5). Describe this transformation fully.

28 Earth Geometry

The Earth is taken to be a true sphere of radius ≈ 6370km.

(ア) Latitude and Longitude

Circles on the Earth

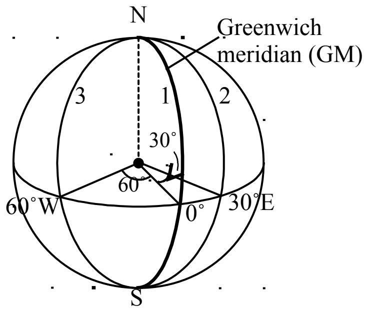


Any circle on the Earth’s surface whose centre is at the centre of the Earth is called a **great circle**.

Other circles are small circles.

The equator and any circle passing through

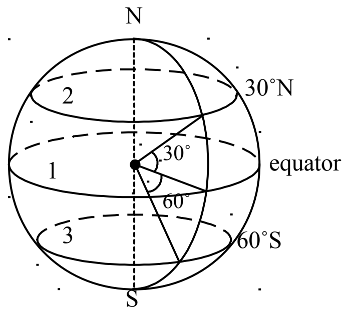
the poles are great circles.
Great circles passing through the poles are called **meridians**.



Longitude: is the angle between the Greenwich meridian, passing through Greenwich (in London), and any other meridian (measured East or West) subtended at the centre of the Earth. In the diagram, (1) is the GM 0°, (2) is a meridian of longitude 30°E and (3) is a meridian of longitude 60°W.

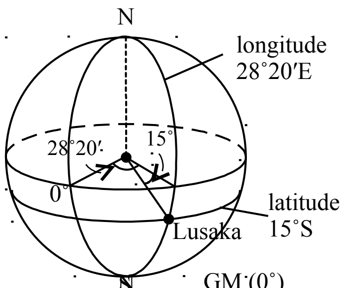
The maximum possible longitude is 180° (E or W).

Latitude: is the angle between the plane of equator and any small circle on the Earth's surface parallel to the equator (measured North or South) subtended at the centre of the Earth.
In the diagram, (1) is the equator 0°, (2) is a small circle of latitude 30°N and (3) is a small circle of latitude 60°S.
The maximum possible latitude is 90° (N or S).

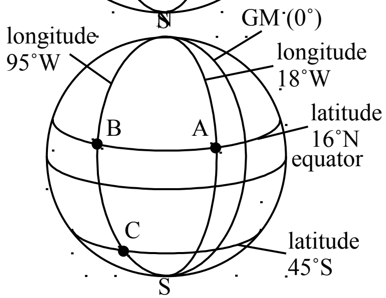


Position of a point: The position of any point on the Earth's surface is given by its latitude and longitude. Latitude and longitude are measured in degrees (°) and minutes (′) with 1° = 60′.

e.g. Lusaka is on latitude 15°S and longitude 28°20′E. This position is given as (15°S, 28°20′E)



Example 1 The diagram shows the points A, B and C on the Earth.



(a) Describe the position of the points A, B and C in the terms of latitude and longitude.
(b) Find the difference in longitude between A and B.
(c) Find the difference in latitude between B and C.

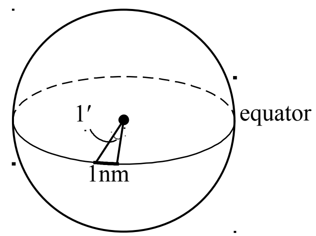
(a) A is on latitude 16°N and longitude 18°W i.e.(16°N, 18°W)
B is on latitude 16°N and longitude 95°W i.e.(16°N, 95°W)
C is on latitude 45°S and longitude 95°W i.e.(43°S, 95°W)
(b) Since the points A and B are on the same sides of the Greenwich Meridian, the difference in latitude is given by: 95° – 18° = 77°
(c) Since the points B and C are on opposite sides of the equator, the difference in longitude is given by: 16° + 45° = 61°

(イ) Distance along Circles of Latitude and Longitude

Nautical mile: is a unit of distance on the Earth's surface used by ships and planes. 1 nautical mile (nm) is 1′ of arc along a great circle (any meridian or the equator). 1 nm ≈ 1852 m = 1.852 km

$1^\circ = 60' \rightarrow 360^\circ = 360 \times 60' = 21\,600'$

∴The circumference of the Earth = 21 600 nm



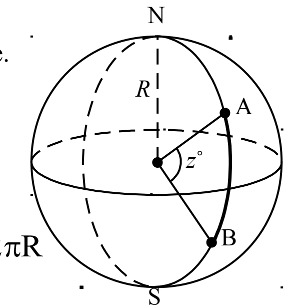
Distance along a circle of longitude

In the diagram, two points A and B lie on the same longitude.
z represents the difference in latitude.

arc AB : circumference of circle = $z^\circ : 360^\circ$

arc AB × 360° = $2\pi R \times z^\circ$

$$\therefore \text{Distance along a circle of longitude} = \frac{z^\circ}{360^\circ} \times 2\pi R$$



Example 2 Given that the radius of the Earth is 6370km, calculate the distance between A(40°N, 50°W) and B(24°30′S, 50°W) in: (a) nautical miles (b) kilometres

(Take π to be 3.142)

(a) The two points lie on the same meridian.

The difference in latitude = 40˚ + 24˚30′ = 64˚30′ = 64 × 60′ + 30′ = 3870′

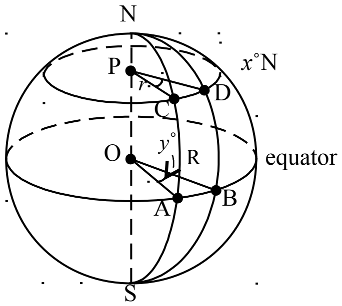
∴The distance = 3870 nm

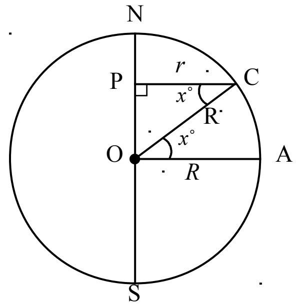
(b) The difference in latitude = 64˚30′ = 64.5˚

∴

$$\text{The distance} = \frac{64.5^\circ}{360^\circ}\times 2\pi R = \frac{64.5^\circ}{360^\circ}\times 2\times 3.142\times 6\,370 = 7172\text{ km}$$

Distance along a circle of latitude





In the diagram, A and B lie on the equator, and C and D lie on the circle of latitude x˚N.

∠AOC = ∠OCP = x˚ because of alternate angles.

In the right-angled triangle OCP, $\cos x^\circ = \dfrac{CP}{OC} = \dfrac{r}{R}$

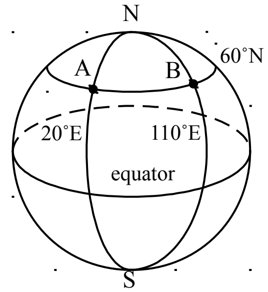
Therefore The radius of a circle of latitude $r = R\cos x^\circ$

∠AOB = ∠CPD = y˚ as these are angles between the same meridians.

arc CD : circumference of circle of latitude x˚N = ∠CPD : 360˚

arc CD × 360˚ = $2\pi r \times y$˚

$$\text{arc CD} = \frac{y^\circ}{360^\circ}\times 2\pi r \quad \text{Now } r = R\cos x^\circ.$$



$$\therefore \text{Distance along a circle of latitude} = \frac{y^\circ}{360^\circ}\times 2\pi R\cos x^\circ$$

Example 3 Point A is on latitude 60˚N, longitude 20˚E and point B is on latitude 60˚N, longitude 110˚E. Given that the radius of the Earth is 6 370km, find the distance along the circle of latitude between A and B in:

(a) nautical miles (b) kilometres (Take π to be 3.142)

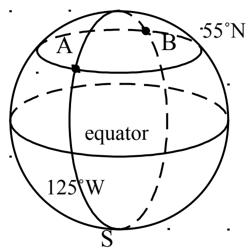
(a) The difference in longitude = 110˚ – 20˚ = 90˚

The distance AB = 90 × 60 × cos60˚ = 2700 nm

(b)

$$\text{The distance AB} = \frac{90^\circ}{360^\circ}\times 2\pi R\cos 60^\circ = \frac{1}{4}\times 2\times 3.142\times 6\,370\times\frac{1}{2} = 5\,004\text{ km}$$

Example 4 The diagram shows that A and B both lie in latitude 55˚N, meridian NAS is 125˚W and meridian NBS is directly opposite NAS.



(a) Write down the positions, using latitude and longitude, of the points: (i) A (ii) B

(b) Calculate the shortest distance AB in nautical miles

(a) (i) A lies on latitude 55˚N and longitude 125˚W, i.e. (55˚N, 125˚W)

(ii) Meridian NBS is 180˚ – 125˚ = 55˚E longitude.

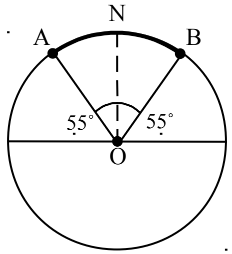
B lies on latitude 55˚N and longitude 125˚W, i.e. (55˚N, 55˚E)

simsiky@gmail.com

(b) Two paths as distance AB can be considered:

(i) via the North pole (along a great circle)

(ii) along the circle of latitude 55˚N



(i) In the diagram on the right,

∠AOB =180˚ – 2 ×55˚ = 70˚

The distance ANB = 70 × 60 = 4200 nm

(ii) The difference in longitude = 180˚

The distance AB along the circle of latitude = 180 × 60 × cos55˚ = 6195 nm

A comparison of the two distances shows that the distance via the North pole is shorter than the distance along the arc of the latitude.

In general,

the shortest distance between two points on the Earth's surface is the length of the arc along the great circle passing through them.

Speed in nautical miles per hour: A speed of 1 nautical mile per hour is 1 **knot**. This is the unit commonly used by ships and planes. (1 knot ≈ 1.852 km/h)

Example 5 A plane flies due south from town A (37.5˚N, 15˚E) to town B (15˚S, 15˚E) in 9hours. Find the speed of the plane in knot.

The difference in latitude = 37.5˚ + 15 = 52.5˚

The distance AB = 52.5 × 60 = 3150 nm

$$\therefore \text{The speed} = \frac{\text{distance}}{\text{time}} = \frac{3150\text{ nm}}{9\text{ h}} = 350\text{ knots}$$

(ウ) Time

The Earth rotates making a complete rotation (360˚) in 24 hours.

Therefore it rotates $\frac{360°}{24} = 15°$ in one hour.

The world is divided into time zones which are determined from the universal time line at the Greenwich meridian. The universal time is called Greenwich Mean Time (GMT).

For every 15˚ moved to east of the Greenwich meridian one hour is gained against GMT and for every 15˚ west of the Greenwich meridian one hour is lost against GMT.

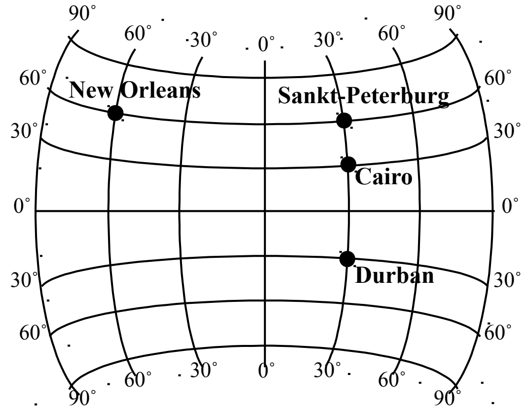
Example 6 Alexandria is on longitude 30˚E. What is the local time in Alexandria if it is 05:00 hours GMT?

Local time is $\frac{30°}{15} = 2$ hours ahead of GMT.

Therefore local time in Alexandria is 05:00 + 2hours = 07:00 hours.

Exercise 20

1 The diagram is an extract from an Atlas of the World map.



(a) Write down the position (in Latitude and Longitude) of the town: (i) New Orleans

(ii) Cairo

(iii) Durban

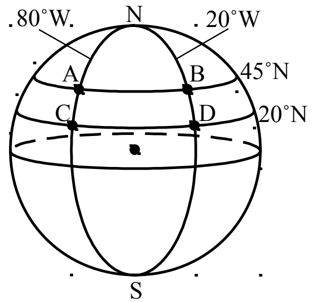
(b) Taking π to be 3.142 and R = 6370km, calculate the distance in kilometres from:

(i) Cairo to Durban (ii) New Orleans to Sankt-Peterburg

(c) When the time in Sankt-Peterburg is 14:00 hours, calculate the time in

(i) Durban (ii) New Orleans

2 The diagram shows positions of four towns A, B, C and D on the Earth's surface. Two jets X and Y are to fly from town A to D. X flies along the circle of latitude 45˚N to B and then due south to D, while Y flies due south to C and then along the circle of latitude 20˚N to D.



(a) Calculate the total distance travelled by each jet in kilometres.

(b) Given that jet X has a speed of 400 knots and assuming that it flies at this speed from A to D, calculate its flight time, giving your answer to the nearest hour.

(c) If is also given that jet Y takes the same time to fly to town D as jet X, calculate the speed of jet Y in knots, giving your answer to the nearest ten.

29 Probability

The probability of an event happening is a measure of how likely that event happens. Probability is measured on a scale from 0 to 1. A value of 0 means it is impossible, while 1 means it is certain. The probability of an event A is usually denoted by P(A).

In general, P(A) + P(not A) = 1. ∴ P(not A) = 1 – P(A)

(ア) Theoretical Probability

Theoretical probability is the probability of an outcome occurring in theory. This probability is based on equally likely outcomes (e.g. coins, dice, cards, etc.). Theoretical probability is expressed as:

$$P(A)=\frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

e.g. If a coin is tossed,

$$\text{The probability of heads} = P(\text{Heads})=\frac{1}{2},\ P(\text{Tails})=\frac{1}{2}.$$

Example 1 A box contains 3 red balls, 2 blue balls and 1 white ball. If a ball is picked at random from the box, what is the probability that it is:

(a) red (b) blue (c) white (d) black (e) not blue

(a) Total number of balls is 3 + 2 + 1 = 6. $P(\text{red})=\frac{3}{6}=\frac{1}{2}$

(b) $P(\text{blue})=\frac{2}{6}=\frac{1}{3}$ (c) $P(\text{white})=\frac{1}{6}$ (d) $P(\text{black})=\frac{0}{6}=0$

(e) $P(\text{not blue})=1-P(\text{blue})=1-\frac{1}{3}=\frac{2}{3}$

(イ) Combined Events

A combined (compound) event involves two or more events

Mutually exclusive events: are sets of events which cannot occur at the same time.

e.g. The outcomes of tossing a coin

∵Either heads or tails turns up. They cannot occur at the same time.

In general, if A and B are mutually exclusive events,

P(A or B) = P(A) + P(B)

This rule is called the addition rule and can be applied to any number of events if they are mutually exclusive.

Example 2 A box contains 3 red balls, 2 blue balls and 1 white ball. If a ball is picked at random from the box, what is the probability that it is red or white?

Total number of balls is 6, 3 of these are red and 1 are white.

$$P(\text{red})=\frac{3}{6},\ P(\text{white})=\frac{1}{6}$$

$$\therefore P(\text{red or white})=\frac{3}{6}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$$

Independent events: are events which have no effect on each other.

e.g. When two dice are rolled, their outcomes are independent.

In general, if A and B are independent events,

P(A and B) = P(A) × P(B)

This rule is called the multiplication rule and can be applied to any number of events if they are independent.

Example 3 A bag contains 2 red beads and 3 white beads. A bead is picked at random from the bag and replaced in the bag. Then a second bead is picked from the same bag. What is the probability that both beads were red?

simsiky@gmail.com

The two events are independent.

$$P(\text{red on 1st picking}) = \frac{2}{5}, \quad P(\text{red on 2nd picking}) = \frac{2}{5}$$

$$\therefore P(\text{both red}) = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$$

<u>Dependent events</u>: Two events are dependent if the first event affects the second event.

e.g. If a bead is picked at random from a bag of red and white beads, and it is not replaced in the bag, the colour of the second picked bead is dependent on the first event.

The multiplication rule can also be used to find the probability of a combination of dependent events.

Example 4 A bag contains 2 red beads and 3 white beads. A bead is picked at random from the bag and is not replaced in the bag. Then a second bead is picked from the same bag. What is the probability that both beads were red?

If the first bead is not replaced, then there are 4 beads remaining in the bag.
And if the first bead was red, then there are only 1 red bead left in the bag.

So $P(\text{red on 1st picking}) = \frac{2}{5}, \quad P(\text{red on 2nd picking}) = \frac{1}{4}$

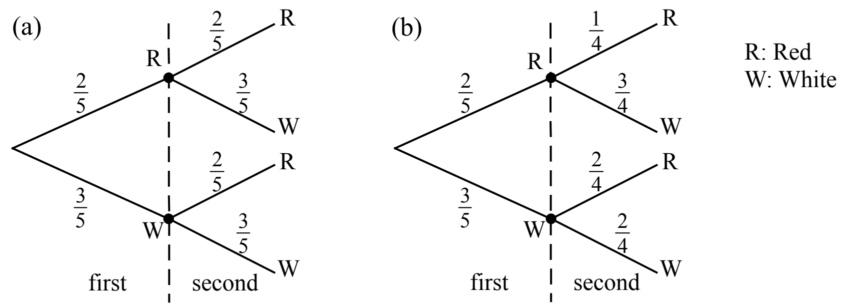
$$\therefore P(\text{both red without replacing}) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$$

<u>Tree diagrams</u>: are diagrams on which the possible outcomes of events are written at the ends of the 'branches'.

e.g. (a) A bag contains 2 red beads and 3 white beads. A bead is picked at random from the bag and replaced in the bag. Then a second bead is picked from the same bag.

(b) A bag contains 2 red beads and 3 white beads. A bead is picked at random from the bag and is not replaced in the bag. Then a second bead is picked from the same bag.

The information above can be shown below.

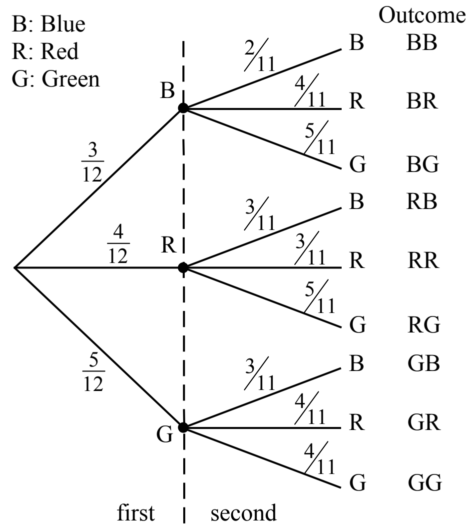


Example 5 A box contains 3 blue marbles, 4 red marbles and 5 green marbles. A marble is taken and is not replaced, and then a second marble is taken.

(a) Draw a tree diagram to show all the possible outcomes.

(b) Find the probability of taking: (i) two green marbles
(ii) one blue marble and one red marble (iii) two marbles of the same colour
(iv) two marbles of different colours (v) no red marble

(a) The tree diagram is shown below.



There are 3 branches for the first taking of a marble and these are marked on the branches. At the end of each of the first branches, there are 3 further branches for the second taking of a marble. The probabilities for these branches are different from the first 3 branches because the first marble is not replaced.

(b) (i) $P(GG) = \frac{5}{12} \times \frac{4}{11} = \frac{5}{33}$

(ii) $P(B \text{ and } R) = P(BR) + P(RB) = \frac{3}{12} \times \frac{4}{11} + \frac{4}{12} \times \frac{3}{11} = \frac{1}{11} + \frac{1}{11} = \frac{2}{11}$

(iii) P(two marbles of the same colour) = P(BB) + P(RR) + P(GG)

$$= \frac{3}{12} \times \frac{2}{11} + \frac{4}{12} \times \frac{3}{11} + \frac{5}{12} \times \frac{4}{11} = \frac{6+12+20}{132} = \frac{38}{132} = \frac{19}{66}$$

(iv) P(two marbles of different colours) = 1 – P(two marbles of the same colour)

$$= 1 - \frac{19}{66} = \frac{47}{66}$$

(v) P(no R) = P(1st not red, 2nd not red)

$$= \left(1 - \frac{4}{12}\right) \times \left(1 - \frac{4}{11}\right) = \frac{8}{12} \times \frac{7}{11} = \frac{14}{33}$$

OR If no R is taken, outcomes are BB, BG, GB and GG

P(no R) = P(BB) + P(BG) + P(GB) + P(GG)

$$= \frac{3}{12} \times \frac{2}{11} + \frac{3}{12} \times \frac{5}{11} + \frac{5}{12} \times \frac{3}{11} + \frac{5}{12} \times \frac{4}{11}$$

$$= \frac{6+15+15+20}{132} = \frac{56}{132} = \frac{14}{33}$$

Exercise 21

1 If a six-sided die is rolled, find the probability that:
(a) 1 turns up (b) an odd number turns up (c) a prime number turns up

2 Two fair coins are tossed. Find the probability of getting a heads and a tails.

3 A bag contains 1 red, 2 yellow and 3 blue beads. A bead is selected and replaced in the bag. Then a second bead is selected What is the probability of getting:
(a) two yellow beads (b) one red bead and one blue bead (c) no red bead
(d) two beads of the same colour (e) two beads of different colours

4 Bag X contains 2 white and 3 black marbles. Bag Y contains 3 white and 2 black marbles. A marble is chosen at random from Bag X and placed in Bag Y without seeing its colour. A marble is now picked from Bag Y. What is the probability that it is white?

5 A box contains 2 white, 5 black and 3 red balls. Two balls are drawn from the bag in succession without replacement.
(a) Draw a tree diagram to show all the possible outcomes.
(b) Find the possibility of getting:
(i) two red balls (ii) one white and one black ball
(iii) one black and one red ball (iv) no black ball
(v) two balls of the same colour (vi) two balls of different colours

30 Statistics

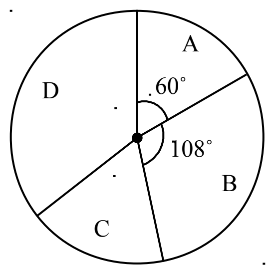
Statistics deals with the collecting, recording, interpreting, illustrating and analysing of data. Statistics are used to make decisions and predict what may happen in the future.

(ア) Presentation

Data can be presented visually in different ways, depending on the nature of the data and what you would like to show.

Pie charts: are circular diagrams in which the angles of sectors represent the frequency (the angles are proportional to the frequency).

Example 1 The pie chart represents 60 pupils.
(a) What angle represents 1 person?
(b) How many people are represented by sector B?
(c) If sector C represents 9 people, find its angle.
(d) How many people does sector D represent?



(a) 360° represents 60 people.

simsiky@gmail.com

$\frac{360°}{60} = 6°$ represents 1 person.

(b) $\frac{108°}{6°} = 18$ people

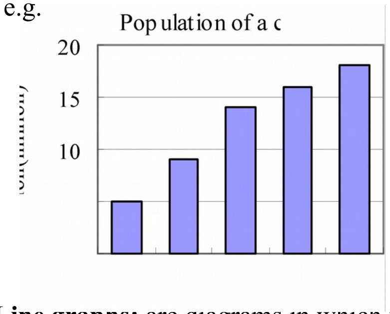
(c) $9 \times 6° = 54°$

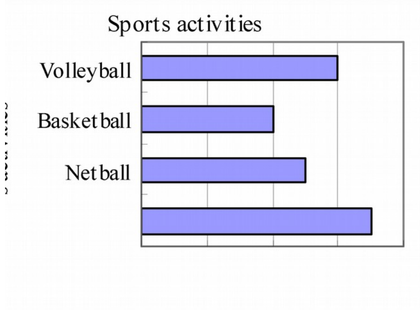
(d) Angle of sector D = 360° – 60° – 108° – 54° = 138° $\therefore \frac{138°}{6°} = 23$ people.

or sector A represents = 60° ÷ 6° = 10 people. So 60 – 10 – 18 – 9 = 23 people

Bar graphs: are diagrams in which the vertical or horizontal bars of equal width represent the frequency (the lengths are proportional to the frequency).

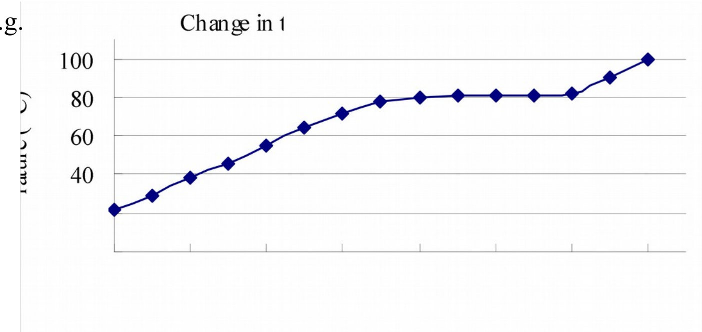
e.g.





Line graphs: are diagrams in which frequencies are plotted and the points are joined by lines.

e.g.



(イ) Measures

Measures of averages (central tendency): There are three kinds of average in statistics – **mean**, **mode** and **median**.

Mean: is the arithmetically calculated average which is found by adding up all the values and dividing by the number of values.

$$\text{Mean} = \frac{\text{Sum of all the values}}{\text{Number of values}}$$

Mode: is the value with the highest frequency.

Median: is the middle value of an ordered set of data. In general,

$\text{position of median} = \frac{n+1}{2}$ where n is the number of values. For an odd number of values, the median is the value in the middle. For an even number of values, the median is the mean of the two middle values.

Example 2 Find (i) the mean, (ii) mode and (iii) median of the numbers:

(a) 2, 3, 5, 6, 6, 9, 10 (b) 5, 2, 1, 3, 6, 5, 5, 2, 3, 4

(a) (i) Mean $= \frac{2+3+5+6+6+9+10}{7} = 5.85 = 5.9$ (to 2sf)

(ii) Mode is 6 (iii) Median is 6

(b) Arrange the numbers in ascending order: 1, 2, 2, 3, 3, 4, 5, 5, 5, 6

(i) Mean $= \frac{1+2+2+3+3+4+5+5+5+6}{10} = 3.6$

(ii) Mode is 5 (iii) Median $= \frac{3+4}{2} = 3.5$

Measures of spread (dispersion): gives how much the data is grouped around the average or spread out. There are several ways of measuring spread.

Range: is the difference between the highest and lowest values.

range = highest value – lowest value

Quartiles: divide the values into four equal parts (this means there are three quartiles). The second quartile (Q_2) is the median and the other two quartiles are called the lower (Q_1) and upper quartile (Q_3). In general, if n is the numbers of values,

$$\text{position of lower quartile}(Q_1) = \frac{n+1}{4}, \text{ position of upper quartile}(Q_3) = \frac{3(n+1)}{4}$$

Interquartile range: is the difference between the upper and lower quartiles.

interquartile range = upper quartile – lower quartile

Example 3 Find the range, the lower quartile, the median, the upper quartile and the interquartile range of the numbers 25, 31, 30, 24, 28, 33, 35, 31, 27, 23, 33.

Arrange the 11 numbers. 23 24 25 27 28 30 31 31 33 33 35

Q_1 Q_2 Q_3

$$\text{position of } Q_1 = \frac{11+1}{4} = 3,\ \text{position of } Q_2 = \frac{11+1}{2} = 6,\ \text{position of } Q_3 = \frac{3(11$$

∴the range = highest value – lowest value = 35 – 23 = 12

the lower quartile = 25

the median = 30

the upper quartile = 33

the interquartile range = upper quartile – lower quartile = 33 – 25 = 8

Percentiles: divide the values into 100 equal parts. Percentiles are used for large amounts of data. In genera, if n is the numbers of values,

$$\text{position of } m\text{th percentile}(P_m) = \frac{m(n+1)}{100}$$

e.g. the 90th percentile

$$P_{90} = \frac{90(n+1)}{100}$$

It can be said that the lower quartile is the 25th percentile, the median is the 50th percentile and the upper quartile is the 75th percentile.

(ㄅ) Frequency Distributions

A frequency distribution is used for large amounts of data.

Frequency tables: show the frequency each value occurs.

Example 4 The data below shows the marks obtained by 40 pupils in a mathematics test. Make a frequency table and find (a) the mean, (b) mode and (c) median of the data.

7 9 5 4 1 3 7 8 6 7 5 7 6 7 8 6 5 4 7 10

3 4 8 6 5 2 5 4 5 1 7 2 6 6 4 9 5 7 6 3

Count the frequencies and then present these marks in a frequency table.

Marks (x)	1	2	3	4	5	6	7	8	9	10
Frequency (f)	2	2	3	5	7	7	8	3	2	1

(a) The mean can also be found by using a frequency table.

Redraw the table with another row for the product (fx) and another column at the end for total. Calculate fx and the total of f and fx.

Marks (x)	1	2	3	4	5	6	7	8	9	10	Total
Frequency (f)	2	2	3	5	7	7	8	3	2	1	40
Product (fx)	2	4	9	20	35	42	56	24	18	10	220

The mean of a frequency distribution is given by the following formula:

$$\text{Mean of a frequency distribution} = \frac{\text{Sum of } fx}{\text{total } f}$$

$$\therefore \text{Mean} = \frac{220}{40} = 5.5$$

(b) The mode can be found easily. Find the marks with highest frequency.

So 7 is the mode.

(c) $\text{position of median} = \frac{40+1}{2} = 20.5$

The 20th number and 21st number both are 6. So the median is 6.

Grouped data: is data put into groups to make it easier to handle when there is a large number of data or the data is widely spread. The groups are called classes and the width of a class is called a **class interval**. In any collection of grouped data, the class intervals are usually all the same but do not have to be.

Example 5 The table below shows the heights of 50 pupils in ascending order. Construct a frequency table using a class interval of 5, beginning with 151–155.

154 155 156 157 157 158 158 159 159 159

159 160 160 160 161 161 162 162 163 163

164 164 165 165 165 165 165 166 166 167

167 167 168 168 168 169 170 170 170 171

171 172 172 172 173 173 174 174 175 176

simsiky@gmail.com

The heights can be grouped into classes having a class interval of 5 as follows.

Heights	151–155	156–160	161–165	166–170	171–175	176–180
Frequency	2	12	13	12	10	1

Mean of grouped data: can be found by using the class centre as the representative for each class. (The class centre is the mid-point of the class interval)

Example 6 Calculate the mean of the grouped data of Example 5.

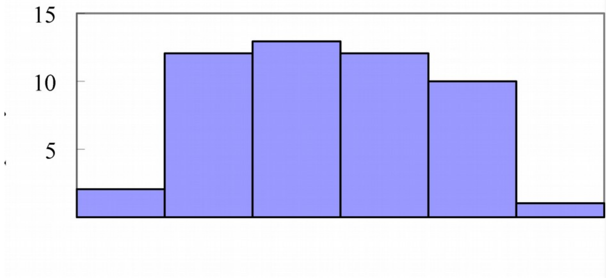
Redraw the table with two more row for the class centre (x) and the product (fx), and another column at the end for total.

Heights	151–155	156–160	161–165	166–170	171–175	176–180	Total
Class centre (x)	153	158	163	168	173	178	
Frequency (f)	2	12	13	12	10	1	50
Product (fx)	306	1896	2119	2016	1730	178	8245

$$\therefore \text{Mean} = \frac{\text{Sum of } fx}{\text{total } f} = \frac{8245}{50} = 164.9$$

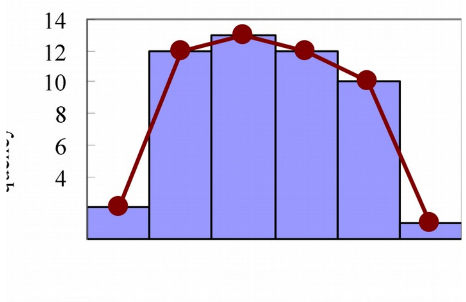
Histograms: are like vertical bar graphs of grouped data in which each frequency is represented by a rectangle, there are no spaces between the bars, the widths of rectangles are proportional to the class interval and the areas of rectangles are proportional to the frequency.

e.g. The diagram on the right shows the histogram for the data of Example 5.

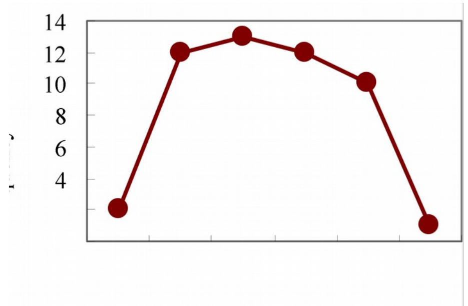


Frequency polygons: are diagrams obtained by joining the **mid-points** of the tops of the bars on a histogram.

e.g. The diagram below shows the frequency polygon of Example 5.



frequency polygon with histogram



frequency polygon without histogram

(工)Cumulative Frequency

A cumulative frequency is the frequencies added to produce a running total. A cumulative frequency is used to make estimates.

Cumulative frequency curves: are diagrams obtained by joining the cumulative frequencies against the **upper class boundaries** with a smooth curve. (The class boundary is the border between two class intervals. The upper class boundary divides a class interval from the one above it.)

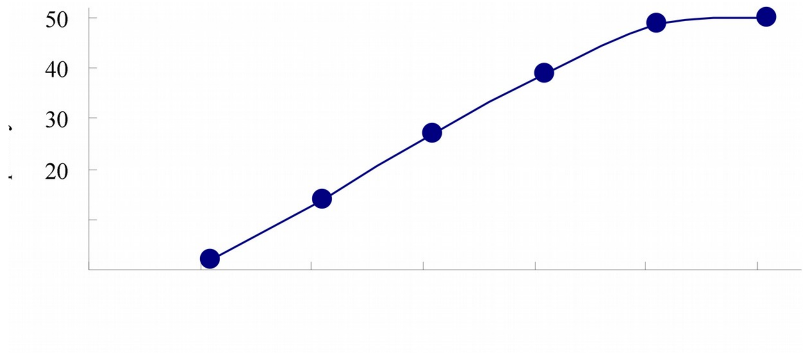
Example 7 Draw a cumulative frequency curve for the data of Example 5.

Redraw the table with two more row for the upper class boundary and the cumulative frequency.

Heights	151–155	156–160	161–165	166–170	171–175	176–180
Upper class boundary	155.5	160.5	165.5	170.5	175.5	180.5
Frequency	2	12	13	12	10	1
Cumulative frequency	2	2 + 12 = 14	14 + 13 = 27	27 + 12 = 39	39 + 10 = 49	49 + 1 = 50

(This table is called a **cumulative frequency table**.)

Plot the cumulative frequencies against the upper class boundaries and join them with a smooth curve.



A cumulative frequency can be used to find out further information about the data such as 'How many pupils scored more than 75%?' or 'How many pupils scored less than 25%?'

Example 8 The table below shows a test marks of 300 pupils.

Marks	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50
Frequency	6	14	49	70	65

Marks	51 – 60	61 – 70	71 – 80	81 – 90	91 –100
Frequency	44	29	13	7	3

(a) Draw a cumulative frequency curve.

(b) Use your graph to estimate: (i) the median (ii) the interquartile range

(c) Pupils scoring 75marks or more were awarded distinctions. Estimate the number of pupils who had distinctions.

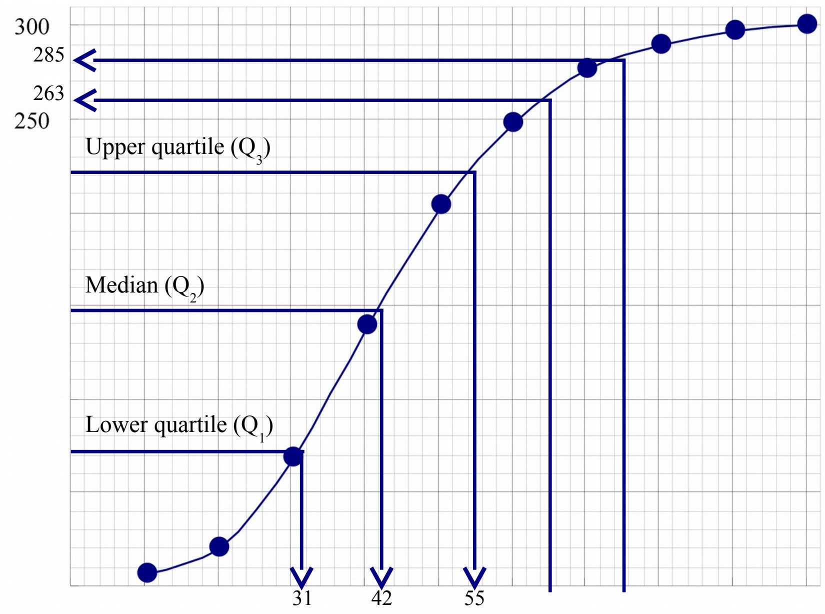
(d) If a pupil is chosen, what is the probability that his marks will be greater than 65 marks?

(a) Construct the cumulative frequency table.

Marks	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50
Frequency	6	14	49	70	65
Cumulative frequency	6	20	69	139	204

Marks	51 – 60	61 – 70	71 – 80	81 – 90	91 –100
Frequency	44	29	13	7	3
Cumulative frequency	248	277	290	297	300

Plot the cumulative frequencies and join them with a smooth curve.



(b) Median = 50th percentile

Upper quartile = 75th percentile Lower quartile = 25th percentile

From the graph,

(i) the median = 42

(ii) the interquartile range = upper quartile – lower quartile = 55 – 31 = 24

(c) The frequency at 75 marks is 285. So 284 pupils were not awarded.

The number of pupils who had distinctions = 300 – 284 = 16

(d) The frequency at 65 marks is 263. So 37 pupils' marks are greater than 65 marks.

So the probability $= \dfrac{37}{300}$

simsiky@gmail.com

Exercise 22

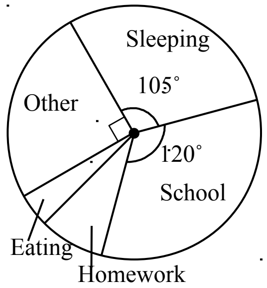
1 The pie chart shows how a pupil spends a day.

(a) What fraction of time is spent for school?

(b) How long is the time for sleeping?

(c) If the time for homework is 2hours, what is the angle of sector 'homework'?

(d) What percentage of time is spent for eating, giving your answer correct to 1dp?



2 Find (a) the mean, (b) mode and (c) median of the numbers:

6, 5, 4, 6, 7, 3, 4, 8, 2, 6

3 The table below shows the marks obtained by a class in a mathematics test.

Marks	0 – 9	10 – 19	20 – 29	30 – 39
Frequency	2	7	23	15

Marks	40 – 49	50 – 59	60 – 69	70 – 79
Frequency	6	3	2	2

(a) Draw a histogram.

(b) Find: (i) the number of pupils who wrote the test (ii) the mean of marks

4 100 pupils were asked the time they took to travel to their school in a city. The results are shown in the following table.

Time in min (x)	$10<x\leq20$	$20<x\leq30$	$30<x\leq40$	$40<x\leq50$
Number of pupils	8	17	23	20

Time in min (x)	$50<x\leq60$	$60<x\leq70$	$70<x\leq80$
Number of pupils	15	11	6

(a) Calculate an estimate of the mean of time taken to travel.

(b) Copy and complete the cumulative frequency table for the time taken to travel.

Time in min (x)	≤20	≤30	≤40	≤50	≤60	≤70	≤80
Number of pupils	8						100

(c) Draw a cumulative frequency curve to illustrate this information.

(d) Use your graph to find: (i) the median of time taken (ii) the interquartile range

(e) If a pupils are chosen at random, find the probability that he will take less than or equal to 35minutes.

31 Sequences

A sequence is a list of numbers which follow a mathematical rule. Each number in a sequence is called a **term** of the sequence. If the rule is not given, it can be worked out from the first few terms.

Example 1 Write down the next two terms in each of the following sequences.

(a) 3, 7, 11, 15, . . . (b) 2, 4, 8, 16, . . .

(a) Each term is obtained by adding 4 to the previous term.

∴The next two terms are: 15 + 4 = 19 and 19 + 4 = 23

(b) Each term is obtained by multiplying the previous term by 2.

∴The next two terms are: 16 × 2 = 32 and 32 × 2 = 64

The *n*th term of a sequence: can be expressed as a function of n. The nth term is denoted by a_n . In general,

-if each next term is obtained by adding a constant to the previous term,

$$a_n = a + (n-1)d$$

where a = the first term, n = the number of terms, d = the common difference

-if each next term is obtained by multiplying the previous term by a constant,

$$a_n = ar^{n-1}$$

where a = the first term, n = the number of terms and d = the common ratio

Example 2 Find the formula for the nth term of following sequences.

(a) 2, 5, 8, 11, . . . (b) 3, 6, 12, 24, . . . (c) 1, 4, 9 , 16, . . .

(a) Each term is obtained by adding 3 to the previous term.

So $a = 2$ and $d = 3$. $\therefore a_n = 2 + (n-1) \times 3 = 3n - 1$

(b) Each term is obtained by multiplying the previous term by 2.

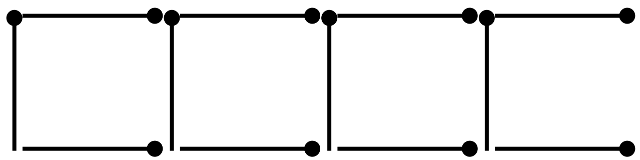
So $a = 3$ and $r = 2$. $\therefore a_n = 3 \times 2^{n-1}$

(c) Each term is obtained by squaring the number of its position.

1st term = $1^2 = 1$, 2nd term = $2^2 = 4$, 3rd term = $3^2 = 9$, . . . $\therefore a_n = n^2$

Applications of sequences

Example 3 Match sticks joined end to end were used to construct squares as shown below. The table shows the relationship between the number of squares (n) and the number of match sticks (m) used.



Number of squares (n)	1	2	3	4	5
Number of match sticks (m)	4	7	10	x	y

(a) Find the value of x and y in the table
(b) Write down the formula of m in terms of n.
(c) Find the number of squares (n) that will be constructed with 151 match sticks (m).

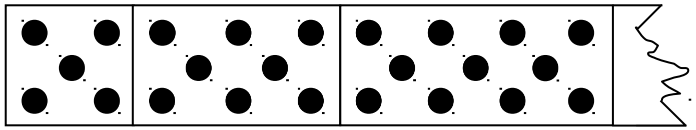
(a) Each term is obtained by adding 3 to the previous term.
Therefore $x = 10 + 3 = 13$ and $y = 13 + 3 = 16$
(b) The first term (a) is 4 and the common difference (d) is 3.
$\therefore m = 4 + (n - 1) \times 3 = 3n + 1$
(c) Substitute 151 for m and find the value of n.
$151 = 3n + 1$ $3n = 150$ $\therefore n = 50$

Exercise 23

1 Find the next two terms: (a) 2, 5, 8, 11, . . . (b) 8, 6 , 4, 2, . . . (c) 1, 10, 100, . . .
(d) 16, –8, 4, –2, . . . (e) 1, 2, 4, 7, 11, . . . (f) 4, 9, 16, 25, . . .

2 Find the nth term: (a) 1, 3, 5, 7, . . . (b) 10, 6, 2, –2, . . (c) 2, 6, 18, 54, . . .
(d) 0, 1, 4, 9, 16, . . . (e) 1, 8, 27, 64, . . .

3 Consider the following pattern.



(a) How many dots are in the 4th rectangle space?
(b) Write down the formula for the number of dots in the nth rectangle.
(c) A given rectangle has 44 dots, what is its position in this pattern?
(d) How many dots are in the 25th rectangle?

simsiky@gmail.com

Answers

Exercise 1

1 (a) 3 (b) 3 (c) 1 (d) 2 (e) 4 (f) 3

2 (a) 7 – 2 + 12 = 17 (b) 12 – 6 = 6 (c) –35 – 8 = –43

3 (a) $2+1-4+\frac{6+4-1}{8}=-1+\frac{9}{8}=\frac{1}{8}$

(b) $2-4+3+\frac{15-5+12}{30}=1+\frac{22}{30}=1\frac{11}{15}$

(c) $\frac{4}{3}\div\frac{7}{6}=\frac{4}{\cancel{3}_1}\times\frac{\cancel{6}^2}{7}=\frac{8}{7}=1\frac{1}{7}$

(d) $\frac{7}{3}\div\left(\frac{5}{2}-\frac{9}{5}\right)-\frac{7}{4}=\frac{7}{3}\div\frac{25-18}{10}-\frac{7}{4}=\frac{\cancel{7}^1}{3}\times\frac{10}{\cancel{7}_1}-$

$=\frac{10}{3}-\frac{7}{4}=\frac{40-21}{12}=\frac{19}{12}\left(=1\frac{7}{12}\right)$

4 42 = 2×3×7, 70 = 2×5×7, 105 = 3×5×7
∴HCF = 7, LCM = 2×3×5×7 = 210

5 $\frac{7}{7+6}\times 455=245\text{ boys}$

6 (a) 135min : 13.5min = 10 : 1

(b) $\frac{8}{12}:\frac{15}{12}:\frac{12}{12}=8:15:12$

7 1cm on the map represents 20 000cm = 0.2km.
(a) 1 : 0.2 = 8 : x ∴x = 0.2×8 = 1.6km

(b) 1 : 0.2 = x : 6 $x=\frac{6}{0.2}=30\text{cm}$

(c) 1cm^2 on the map represents 0.2×0.2 = 0.04km^2.
1 : 0.04 = 200 : x ∴x = 0.04×200 = 8km^2

(d) 1 : 0.04 = x : 3 $x=\frac{3}{0.04}=75\text{ cm}^2$

Exercise 2

1 Cost = 10×45 000 = K450 000
Income = 9×52 000 + 1×40 000 = K508 000
Profit/Loss = 508 000 – 450 000 = K58 000 (profit)

2 (a(a) $\text{Discount}=\frac{5}{100}\times 120\,000=6\,000$

120 000 – 6 000 = K114 000

(b) $\text{original price}\times\frac{100-5}{100}=\text{K}28\,500$

$\text{original price}=28\,500\times\frac{100}{95}=\text{K}30\,000$

3 $I=\dfrac{1\,200\,000\times 9.5\times\frac{8}{12}}{100}=\text{K}76\,000$

A = 1 200 000 + 76 000 = K1 276 000

4 $\text{Amount}=832\,000=P+\frac{P\times 7\times 4}{100}$

$\therefore P=\frac{832\,000\times 100}{128}=\text{K}650\,000$

5 $R=\frac{I\times 100}{PT}=\frac{120\,000\times 100}{250\,000\times 6}=8\%$

Exercise 3

1 (a) $x^{\frac{1}{2}-\frac{1}{3}}=x^{\frac{1}{6}}$ (b) $x^{-2+\frac{1}{2}-\left(-\frac{3}{2}\right)}=x^0=1$

2 (a) $\frac{1}{(-5)^3}=-\frac{1}{125}$ (b) $\left(\frac{2}{1}\right)^2=4$

(c) $5^{6-1-3}\times 1=5^2=25$

(d) $4^{2.5-1}=4^{1.5}=4^{\frac{3}{2}}=\left(\sqrt{4}\right)^3=2^3=8$

or $\left(2^2\right)^{\frac{3}{2}}=2^{2\times\frac{3}{2}}=2^3=8$

(e) $\left(\sqrt[3]{-\frac{8}{27}}\right)^2=\left(-\frac{2}{3}\right)^2=\frac{4}{9}$

or $\left\{\left(-\frac{2}{3}\right)^3\right\}^{\frac{2}{3}}=\left(-\frac{2}{3}\right)^2=\frac{4}{9}$

(f) $\frac{1}{25^{\frac{1}{2}}}=\frac{1}{\sqrt{25}}=\frac{1}{5}$

(g) $16^{\frac{3}{4}}=\left(\sqrt[4]{16}\right)^3=2^3=8$ or $\left(2^4\right)^{\frac{3}{4}}=2^3=8$

3 (a) 7.03×10^2 (b) 8.4052×10^3 (c) 7.2×10^{-2}
(d) 3.75×10^{-4}

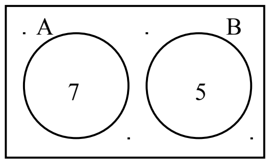
4 (a) 3.0 (b) 2.97 (c) 2.974

5 (a) 40 (b) 41 (c) 41.0 (d) 40.97

Exercise 4

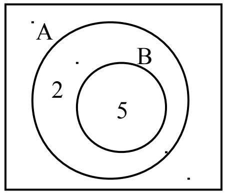
1 E = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, A = {2, 3, 5, 7} B = {1, 3, 5, 7, 9}, C = {3, 6, 9}
(a) A' = {1, 4, 6, 8, 9} (b) A ∩ B = {3, 5, 7}
(c) B' = {2, 4, 6, 8}, C' = {1, 2, 4, 5, 7, 8}
∴B' ∩ C' = {2, 4, 8}
(d) B ∩ C = {3, 9} ∴A ∪ (B ∩ C) = {2, 3, 5, 7, 9}

2 If A and B don't intersect, n(A∪B) is

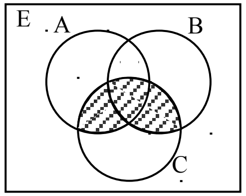


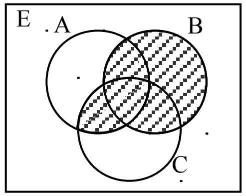
largest and n(A∩B) is smallest. The largest value of n(A∪B) is 7 + 5 = 12 and the smallest value of n(A∩B) is 0.

If B⊂A (Fig 4.2), n(A∪B) is smallest and n(A∩B) is largest. The smallest value of n(A∪B) is 7 and the largest value of n(A∩B) is 5.

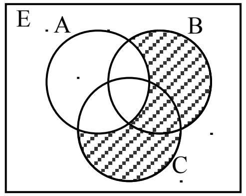


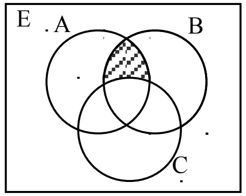
3 (a)



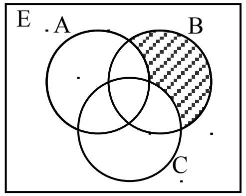


(c)

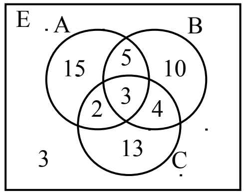




(e)



4



Exercise 5

1 (a) $6x^2 - 2xy$ (b) $5x - 19y$ (c) $a^2 + 3a - 10$
(d) $3x^2 - 2xy - 8y^2$ (e) $4a^2 - 12ab + 9b^2$

(f) $4x^2 - 4 + \dfrac{1}{x^2}$

2 (a) $3x(x - 2)$
(b) $3a(c - 2d) - 2b(c - 2d) = (3a - 2b)(c - 2d)$
(c) $(x + 1)(x + 2)$ (d) $(x + 3)^2$ (e) $(x - 7)(x + 3)$

p	2
s	3
f	1, 2

p	9
s	6
f	3, 3

p	–21
s	–4
f	–7, 3

(f) $2x^2 - 5x - 3$
$= 2x^2 - 6x + x - 3$
$= 2x(x - 3) + 1(x - 3)$
$= (x - 3)(2x + 1)$

p	–6
s	–5
f	–6, 1

(g) $12x^2 - 5xy - 2y^2$
$= 12x^2 - 8xy + 3xy - 2y^2$
$=4x(3x - 2y) + y(3x - 2y)$
$=(3x - 2y)(4x + y)$

p	$-24y$
s	$-5y$
f	$-8y, 3y$

(h) $(7 + 4x)(7 - 4x)$ (i) $(ab + 2c)(ab - 2c)$
(j) $s(t^2 - 4) + 2(t^2 - 4) = (s + 2)(t + 2)(t - 2)$

3 (a) $\dfrac{1}{z}$ (b) $\dfrac{(3x-2)(x+4)}{(3x-2)(2x-3)} = \dfrac{x+4}{2x-3}$

(c) $\dfrac{-(a-2)}{(a+2)(a-2)} = \dfrac{-1}{a+2}$ (d) $\dfrac{1+2\times 2}{2a} = \dfrac{5}{2a}$

(e) $\dfrac{1\times(x-y)+1\times(x+y)}{(x+y)(x-y)} = \dfrac{2x}{(x+y)(x-y)}$

(f) $\dfrac{1}{x-2} - \dfrac{1}{(2x-3)(x-2)} = \dfrac{2x-3-1}{(2x-3)(x-2)}$

$= \dfrac{2(x-2)}{(2x-3)(x-2)} = \dfrac{2}{2x-3}$

(g) $\dfrac{2(x-y)}{9} \times \dfrac{3}{-(x-y)} = -\dfrac{2}{3}$

(h) $\dfrac{x+1}{(x-3)(x-4)} \times \dfrac{2(x-4)}{(x+1)^2} = \dfrac{2}{(x-3)(x+1)}$

(i) $\dfrac{(x+y)(x+3y)}{xy} \times \dfrac{y}{x+y} = \dfrac{x+3y}{x}$

Exercise 6

1 (a) $R = \dfrac{100I}{PT}$ (b) $r = \sqrt[3]{\dfrac{4V}{3\pi}}$

(c) $(2p+1)t = q \quad \therefore t = \dfrac{q}{2p+1}$

(d) $y^2 = \dfrac{x+1}{3x-2} \quad y^2(3x-2) = x+1$

$(3y^2 - 1)x = 2y^2 + 1 \quad \therefore x = \dfrac{2y^2+1}{3y^2-1}$

2 (a) $x = \dfrac{1}{2}$ (b) $x = 0$ (c) $x = 10$ (d) $x = \dfrac{2}{3}$

3 (a) $x = 3, y = 6$ (b) $x = 6, y = -2$

(c) $x = \dfrac{4}{5}$, $y = -\dfrac{2}{5}$ (d) $x = -1, y = 2$

4 (a) $x(x - 3) = 0$ $x = 0, 3$
(b) $(x + 2)(x + 3) = 0$ $x = -2, -3$

simsiky@gmail.com

(c) $(x-3)^2=0 \quad x=3$

(d) $(2x+1)(x-3)=0 \quad x=-\frac{1}{2}, 3$

(e) $x=\frac{1\pm\sqrt{13}}{6}=\frac{1\pm 3.606}{6}=0.77, -0.43$

(f)

$x=\frac{-1\pm\sqrt{41}}{4}=\frac{-1\pm 6.403}{4}=1.35, -1.85$

5 (a) $x=(-27)^{\frac{1}{3}}=-3$ (b) $x=4^{\frac{3}{2}}=2^3=8$

(c) $x=\left(\frac{1}{16}\right)^{\frac{1}{4}}=\left(\frac{1}{2}\right)^{4\times\frac{1}{4}}=\frac{1}{2}$ (d)

$x=8^{-\frac{1}{3}}=2^{-1}=\frac{1}{2}$

(e) $2^x=2^4 \therefore x=4$ (f) $10^x=10^0 \therefore x=0$

(g) $5^{2x}=5^{-1} \quad 2x=-1 \quad \therefore x=-\frac{1}{2}$

(h) $2^{-4x}=2^3 \quad -4x=3 \quad \therefore x=-\frac{3}{4}$

6 (a) $f(-3)=\frac{2\times(-3)-3}{-3}=3$

(b) $5=\frac{2x-3}{x} \quad 5x=2x-3 \quad \therefore x=-1$

(c) $xy=2x-3 \quad x=\frac{-3}{y-2} \quad \therefore f^{-1}(x)=\frac{-3}{x-2}$

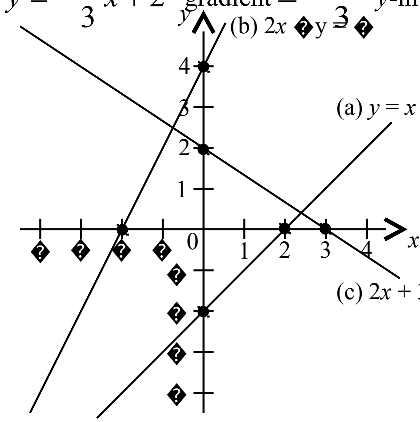
(d) $f^{-1}(-4)=\frac{-3}{-4-2}=\frac{1}{2}$

Exercise 7

1 (a) gradient = 1 y-intercept = –2

(b) $y=2x+4$ gradient = 2 y-intercept = 4

(c) $y=-\frac{2}{3}x+2$ gradient $=-\frac{2}{3}$ y-intercept = 2



2 (a)

x	–3	–2	–1	0	1	2	3
y	18	8	2	0	2	8	18

(b) y-intercept = (0, 10)

$-x^2+3x-10=0$

$(x-5)(x+2)=0$

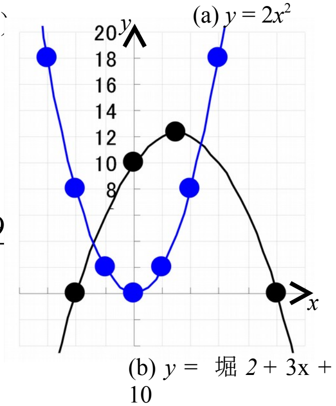
∴ x-intercept

= (5, 0) and (–2, 0)

$y=-\left(x-\frac{3}{2}\right)^2+\frac{49}{4}$

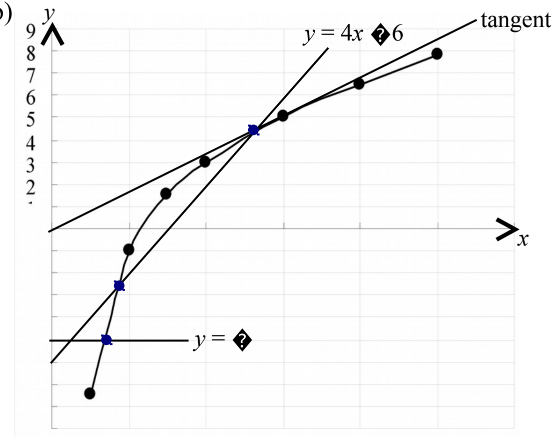
∴ The turning point

$=\left(\frac{3}{2}, \frac{49}{4}\right)$



3 (a) $a=4-\frac{6}{5}+5=7.8$

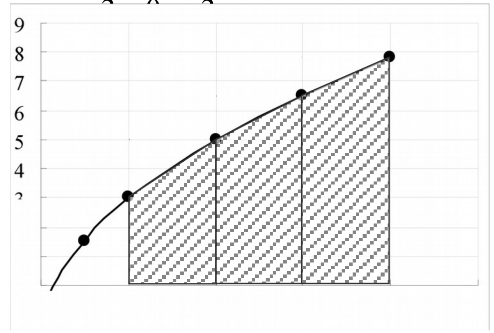
(b)



(c) The tangent passes through the origin.

gradient $=\frac{5-0}{2-0}=\frac{5}{2}$

(d)



Draw three trapeziums in which each height is 1 unit.

Area

$$= \frac{1}{2}(3+5)\times 1 + \frac{1}{2}(5+6.5)\times 1 + \frac{1}{2}(6.5+7.8)\times 1$$

= 16.9 unit^2

(e) From the graph, $x = 0.7$

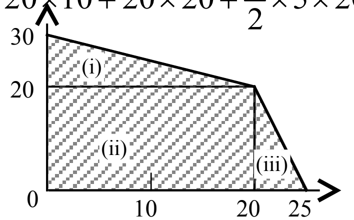
(f) $4 + 6 - \frac{6}{x} + x - 4x = 0 \quad \therefore 4 - \frac{6}{x} + x = 4x - 6$

Draw the line $y = 4x - 6$. From the graph, $x = 0.9, 2.6$

4 (a) $\text{deceleration} = \frac{20-30}{20} = -0.5\ \text{m/s}^2$

(b) total distance = area of (i) + area of (ii) + area of (iii)

$$= \frac{1}{2}\times 20\times 10 + 20\times 20 + \frac{1}{2}\times 5\times 20 = 550\ \text{m}$$



(c)

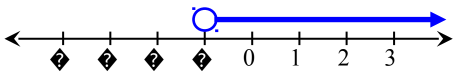
$$\text{average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{550}{25} = 22\ \text{m/s}$$

(d) $\text{deceleration} = \frac{0-20}{5} = -4\ \text{m/s}^2$

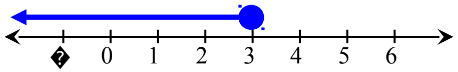
speed = 20 + (–4)×2 = 12 m/s

Exercise 8

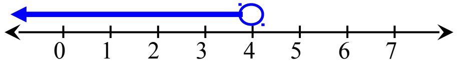
1 (a) $x > -1$



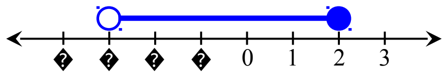
(b) $x \le 3$



(c) $x < 4$



(d) $-3 < x \le 2$



2 $x + 2 < 7 \Rightarrow x < 5 \quad 7 < 2x + 1 \Rightarrow 3 < x$

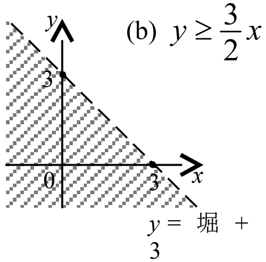
$3 < x < 5 \;\therefore x = 4$

3 (a) smallest x + smallest y = –2 + (–6) = –8

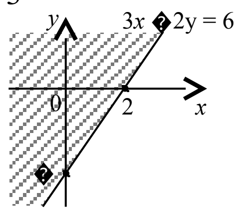
(b) largest x – smallest y = 3 – (–6) = 9

(c) $y = -5$

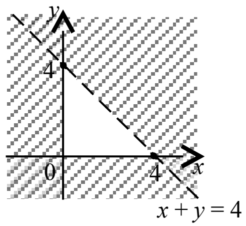
4 (a)



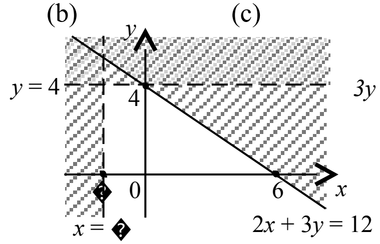
(b) $y \ge \frac{3}{2}x - 3$

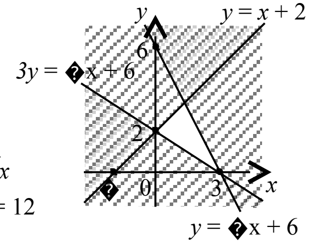


5 (a) $x = 0 \Rightarrow y$-axis, $y = 0 \Rightarrow x$-axis



(b) (c)

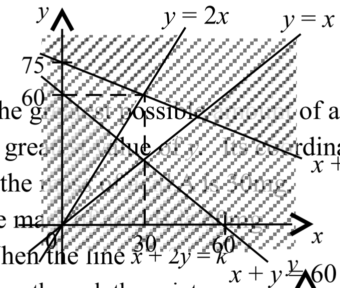




6 (a) (i) $x + y \ge 60$ (ii) $y \ge x$ (iii) $y \le 2x$

(iv) $x + 2y \le 150$

(b)



(c) The g... possib... of acid B means that the grea... lue ... rdinates are (30, 60).

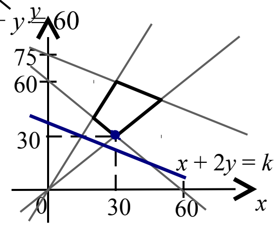
So the ...

The ma...

(d) When the line $x + 2y = k$ passes through the point (30, 30), the volume becomes smallest.

So the volume is

30×1 + 30×2 = 90mm^3



Exercise 9

1 $6\times 12 = x\times 8 \quad \therefore x = 9$ people

2

$$y = kx^2 \quad k = \frac{y}{x^2} = \frac{8}{2^2} = 2 \quad \therefore y = 2x^2 = 2\times 5^2 = 50$$

3 $y = \frac{k}{x-2} \qquad k = y(x-2) = 2\times(5-2) = 6$

simsiky@gmail.com

$\therefore y = \dfrac{6}{x-2} = \dfrac{6}{8-2} = 1$

4 (a) $y = k\dfrac{x^2}{z}$ $k = \dfrac{yz}{x^2} = \dfrac{5\times 3}{1^2} = 15$ $\therefore y = \dfrac{15x^2}{z}$

(b) $y = \dfrac{15\times 2^2}{5} = 12$

(c) $x^2 = \dfrac{yz}{15} = \dfrac{6\times 10}{15} = 4$ $\therefore x = \pm 2$

Exercise 10

1 $3x + 90^\circ + 114^\circ = 360^\circ$ $\therefore x = 52^\circ$

2 (a) $\angle ABD = \angle BDE = 46^\circ$

(b) $\angle EBC = 180^\circ - \angle BEF = 102^\circ$

(c) $\angle BEF = \angle DBE + \angle BDE = 78^\circ$

(or $= 180^\circ - \angle ABE$)

3 $(10-2)\times 180^\circ = 1440^\circ$

4 Each exterior angle $= 180^\circ - 135^\circ = 45^\circ$

$\dfrac{360^\circ}{n} = 45^\circ \therefore n = \dfrac{360^\circ}{45^\circ} = 8$ or

$\dfrac{(n-2)\times 180^\circ}{n} = 135^\circ$

5 (a) $\angle CED = \angle EBA = 60^\circ$ ∵corresponding angles

$\angle BED = 180^\circ - \angle CED = 180^\circ - 60^\circ = 120^\circ$

(b) $\angle BAD = \angle BED = 120^\circ$ ∵opposite angles

$\angle BAC = 120^\circ - 35^\circ = 85^\circ$

(c) $\angle ADF = \angle ABE = 60^\circ$ ∵opposite angles

$\angle CFD = \angle FAD + \angle ADF = 35^\circ + 60^\circ = 95^\circ$

6 each interior angle $= \dfrac{(6-2)\times 180^\circ}{6} = 120^\circ$

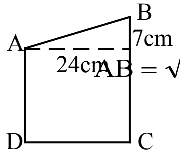
$\angle FAB = \dfrac{180^\circ - 120^\circ}{2} = 30^\circ$

$\therefore \angle BAE = 120^\circ - 30^\circ = 90^\circ$

7 $x = \sqrt{6^2 - 3^2} = \sqrt{27} \approx 5.20\text{cm}$

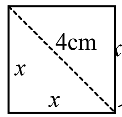
$y = \sqrt{x^2 - 2^2} = \sqrt{27-4} = \sqrt{23} \approx 4.80\text{cm}$

8



$AB = \sqrt{24^2 + 7^2} = \sqrt{625} = 25\text{cm}$

9

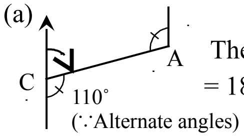


$^2 + x^2$ $16 = 2x^2$

$\sqrt{8} \approx 2.83\text{cm}$

10 $AB = \sqrt{(10-4)^2 + (14-6)^2} = \sqrt{100} = 10$ units

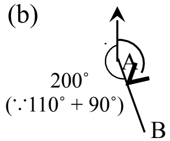
11 (a)



The bearing of A from C

$= 180^\circ - 110^\circ = 070^\circ$

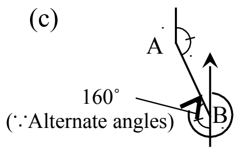
(b)



The bearing of B from A

$= 360^\circ - 200^\circ = 160^\circ$

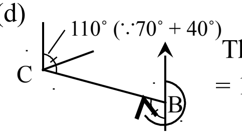
(c)



The bearing of A from B

$= 180^\circ + 160^\circ = 340^\circ$

(d)



The bearing of C from B

$= 180^\circ + 110^\circ = 290^\circ$

Exercise 11

1 (a) $x = \dfrac{1}{2}\times 70^\circ = 35^\circ$

$y = \dfrac{1}{2}\times(360^\circ - 70^\circ) = 145^\circ$

(b) $x = 35^\circ + 40^\circ = 75^\circ$

(c) $x = 180^\circ - 100^\circ = 80^\circ$ $y = 80^\circ + 30^\circ = 110^\circ$

(d) $x = 2\times 47^\circ = 94^\circ$ $y = \dfrac{180^\circ - 94^\circ}{2} = 43^\circ$

(e) $x = 180^\circ - 2\times 70^\circ = 40^\circ$

(f) $x = 180^\circ - 40^\circ = 140^\circ$

2 (a) $\angle BDT = \dfrac{180^\circ - 50^\circ}{2} = 65^\circ$

$\therefore \angle BDC = 65^\circ - 41^\circ = 24^\circ$

(b) $\angle CEB = \angle CDT + \angle DTE = 41^\circ + 50^\circ = 91^\circ$

(c) $\angle BAD = \angle BCE = 65^\circ$

(d) $\angle DBC = \angle CDT = 41^\circ$

3 (a) $\angle REC = \angle EAC = 54^\circ$

(b) $\angle BEC = 180^\circ - 90^\circ - 54^\circ = 36^\circ$

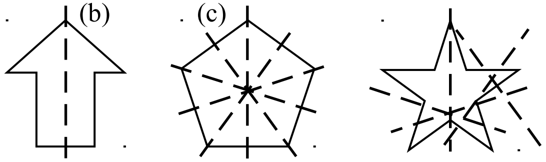
(c) $\angle SEA = \angle EBA = 180^\circ - 68^\circ - 54^\circ = 58^\circ$

(d) $\angle CDE = 180^\circ - 54^\circ = 126^\circ$

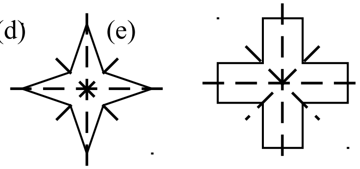
$\therefore \angle CED = \dfrac{180^\circ - 126^\circ}{2} = 27^\circ$

Exercise 12

1 (a) (b) (c)



(d) (e)



2

	A	D	E	F	H	M	S	T	Y
No. of lines	1	1	1	0	2	1	0	1	1
Order	1	1	1	1	2	1	2	1	1

3

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
No. of lines	6	8	0	1	2	1	∞
Order	6	8	2	1	2	1	∞

4 (a) (i) 4 (ii) 0 (b) (i) 4 (ii) 4

Exercise 13

1 $\frac{AD}{DB} = \frac{AE}{EC} \quad \frac{3}{6} = \frac{x}{5} \quad \therefore x = 2.5\text{ cm}$

2 $AD = \sqrt{\frac{4}{9}}AB = \frac{2}{3}AB = 4\text{ cm}$

3 (a) $\triangle ABC$ is similar to $\triangle DBA$.

$\frac{DB}{AB} = \frac{DA}{AC} \quad \frac{5}{x} = \frac{4}{6} \quad \therefore x = 7.5\text{ cm}$

(b)

$$\text{Area of } \Delta PQS = \left(\frac{2}{3}\right)^2 \times \text{area of } \Delta PQR = 8\text{ cm}^2$$

Exercise 14

1 (a) $10 + 8 + 8 + 10 + \frac{106°}{360°} \times 2 \times 3.142 \times 10$

= 36+ 18.50 = 54.5 m (to 3sf)

(b) $2 \times \frac{1}{2} \times (4 + 10) \times 8 + \frac{106°}{360°} \times 3.142 \times 10^2$

= 112 + 92.51 = 204.5 m^2 (to 3sf)

2 Shaded area = area of sector OABC – area of $\triangle OAC$

$$= \frac{90°}{360°} \times \frac{22}{7} \times 7^2 - \frac{1}{2} \times 7^2$$

= 38.5 – 24.5 = 14 cm^2

3 (a) $V = \pi r_{rod}^{\ 2} h = \frac{22}{7} \times 0.7^2 \times 40 = 61.6\text{ cm}^3$

(b) A = area of circle of blade – area of hole

$$= \pi r_{blade}^{\ 2} - \pi r_{rod}^{\ 2} = \pi(r_{blade} + r_{rod})(r_{blade} - r_{rod})$$

$$= \frac{22}{7} \times (7 + 0.7) \times (7 - 0.7) = 152.46\text{ cm}^2$$

(c) V = area×height = 152.46×2 = 304.92 cm^3

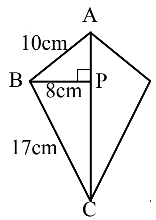
(d) Mass = Density×Volume

Total mass = mass of blade + mass of rod

= 3×304.92 + 6×61.6

= 1284.36g

4 (a) $AP = \sqrt{10^2 - 8^2} = 6\text{ cm}$



$PC = \sqrt{17^2 - 8^2} = 15\text{ cm}$

$\therefore$AC = AP + PC = 21 cm

(b) A = 2×area of $\triangle ABC$

$$= 2 \times \frac{1}{2} \times 21 \times 8 = 168\text{ cm}^2$$

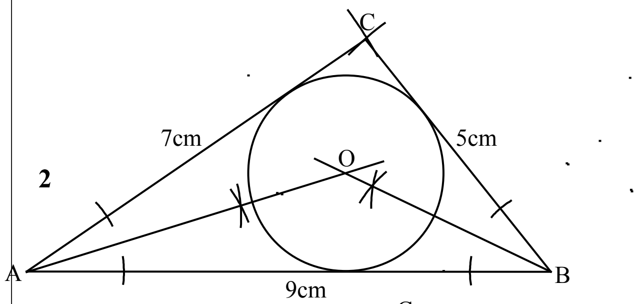
(c) V = area of cross-section×length

= 168×25 = 4200 cm^3

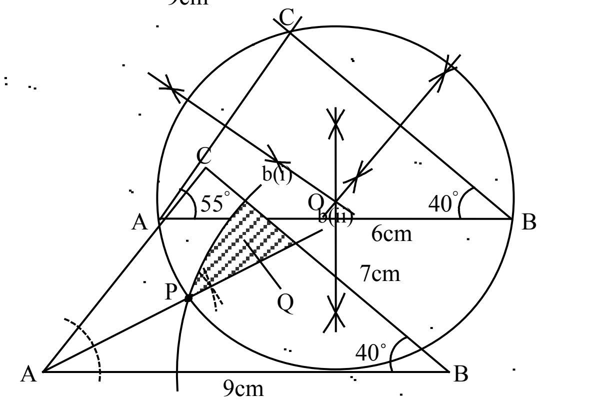
Exercise 15

1

2



3



simsiky@gmail.com

Exercise 16

1 (a) $DB = \sqrt{4.2^2 + 2.5^2} = 4.89$ m

(b) $\tan\angle ECB = \frac{EB}{CE} = \frac{4.2}{6} = 0.7$

$\therefore \angle ECB = \tan^{-1}0.7 = 35.0°$

(c)

$\sin\angle CAE = \frac{CE}{AC}$ $\therefore AC = \frac{6}{\sin 42°} = 8.97$ m

2

$BD = AB \times \tan\angle DAB = 20 \times \tan 40° = 16.78$ cm

$BC = 20 \times \tan 25° = 9.32$ cm

CD = BD – BC = 16.78 – 9.32 =7.46 = 7.5 cm

3 $BC = \sqrt{5^2 - 3^2} = 4$ cm

$\therefore \cos\theta = -\cos(180° - \theta) = -\frac{4}{5}$

4 Area of segment = Area of sector – Area of triangle

$A = \frac{54°}{360°} \times \pi \times 8^2 - \frac{1}{2} \times 8 \times 8 \times \sin 54°$

= 30.16 – 25.89 = 4.27 cm^2

5 (a) $x = 4\times\tan 55° = 5.71$cm

(b) $x = \frac{10}{\cos 35°} = 12.2$ m

(c) $x^2 = 3^2 + 5^2 - 2\times3\times5\times\cos 68° = 22.76$

$\therefore x = 4.77$ cm

$\sin\theta = \frac{5 \times \sin 68°}{x} = 0.9719$

$\therefore \theta = \sin^{-1}0.9719 = 76.4°$

(d) $x^2 = 5^2 + 9^2 - 2\times5\times9\times\cos 130° = 131.85$

$\therefore x = 11.5$ cm

$\sin\theta = \frac{9 \times \sin 130°}{x} = 0.60$ $\therefore \theta = \sin^{-1}0.60$ =36.9°

(e) $\cos\theta = \frac{10^2 + 8^2 - 7^2}{2 \times 10 \times 8} = 0.71875$

$\therefore \theta = \cos^{-1}0.71875 = 44.0°$

$x^2 = 4^2 + 5^2 - 2\times4\times5\times\cos\theta = 12.25$ $\therefore x = 3.5$ cm

Exercise 17

1 (a) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 5 & -5 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 0 \\ 1 & 13 \end{pmatrix}$ (d) $\begin{pmatrix} 12 & -11 \end{pmatrix}$ (e) $\begin{pmatrix} -6 & 4 & 16 \\ 8 & 10 & 12 \end{pmatrix}$

2 (a) $\begin{pmatrix} 8 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} -3 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & 6 \\ 2 & 8 \end{pmatrix}$

(e) $\begin{pmatrix} 8 & -8 & 10 \\ 2 & 5 & -8 \end{pmatrix}$

3 (a) $\begin{pmatrix} 0 & 0 \\ 5 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 6 \\ -1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 4 \\ 11 & 14 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 2 \\ 5 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} -7 & -10 \\ 8 & 12 \end{pmatrix}$ (f) $\begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} - \begin{pmatrix} -3 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 12 \\ 13 & 22 \end{pmatrix}$

4 AB $= \begin{pmatrix} 4 & 1+2b \\ 2a-1 & a-b \end{pmatrix}$, BA $= \begin{pmatrix} 2+a & 4-b \\ 1+ab & 2-b \end{pmatrix}$

$\therefore a = 2, b = 1$

5 $2a\times1 - (-4)\times a = 12$ $\therefore a = 2$

6 det $= x(x + 8) - 3(x + 2) = 0$

$(x + 6)(x - 1) = 0$ $\therefore x = -6, 1$

7 (a) $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 5 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 16 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-7}\begin{pmatrix} -2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 14 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

Exercise 18

1 (a) $\underline{a} + \underline{b} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ (b) $\underline{a} - \underline{b} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ (c) $3\underline{a} + 2\underline{b} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$

(d) $|\underline{a}| = \sqrt{3^2 + 2^2} = \sqrt{13}$ (e) $|\underline{b}| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$

(f) $|\underline{a} + \underline{b}| = \sqrt{2^2 + 6^2} = \sqrt{40}$

(g) $|\underline{a}-\underline{b}|=\sqrt{4^2+(-2)^2}=\sqrt{20}$

2 (a) (i) $\overrightarrow{AB}=-\underline{a}+\underline{b}$

(ii) $\overrightarrow{OM}=\overrightarrow{OA}+\frac{1}{3}\overrightarrow{AB}=\underline{a}+\frac{1}{3}(-\underline{a}+\underline{b})=\frac{2}{3}\underline{a}+\frac{1}{3}\underline{b}$

(b) $\overrightarrow{AN}=-\overrightarrow{OA}+\overrightarrow{ON}=-\underline{a}+\frac{1}{2}\underline{b}$

(c)

$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}=\overrightarrow{OA}+h\overrightarrow{AN}=\underline{a}+h\left(-a+\frac{1}{2}\underline{b}\right)$

$=(1-h)\underline{a}+\frac{1}{2}h\underline{b}$

3 (a)

(i) $\overrightarrow{OE}=\frac{1}{3}\underline{a}$ (ii) $\overrightarrow{BE}=-\overrightarrow{OB}+\overrightarrow{OE}=-\underline{b}+\frac{1}{3}\underline{a}$

(b) (i) $\overrightarrow{BF}=k\left(-\underline{b}+\frac{1}{3}\underline{a}\right)$

(ii) $\overrightarrow{OF}=\overrightarrow{OB}+\overrightarrow{BF}=\frac{1}{3}k\underline{a}+(1-k)\underline{b}$

(c) $\overrightarrow{OF}=h\overrightarrow{OC}=h(\underline{a}+\underline{b})$

(d) $\frac{1}{3}k=h,\ 1-k=h \quad \therefore k=\frac{3}{4},\ h=\frac{1}{4}$

(e) $\overrightarrow{OF}=\frac{1}{4}\overrightarrow{OC}$ ∴OF:FC = 1:3

4 (a) $\overrightarrow{BE}=k\overrightarrow{CB}=k\overrightarrow{OA}=k\underline{a}$

(b)

$\overrightarrow{DE}=h\overrightarrow{OD}=h\left(\overrightarrow{OA}+\overrightarrow{AD}\right)=h\left(\overrightarrow{OA}+\frac{2}{3}\overrightarrow{AB}\right)$

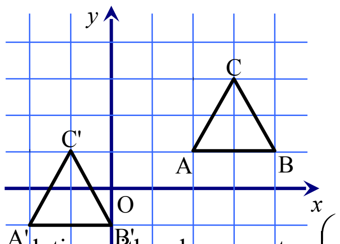
$=h\left(\overrightarrow{OA}+\frac{2}{3}\overrightarrow{OC}\right)=h\left(\underline{a}+\frac{2}{3}\underline{c}\right)$

(c) $\overrightarrow{DE}=\overrightarrow{DB}+\overrightarrow{BE}=\frac{1}{3}\overrightarrow{AB}+\overrightarrow{BE}=\frac{1}{3}\underline{c}+k\underline{a}$

$h\left(\underline{a}+\frac{2}{3}\underline{c}\right)=k\underline{a}+\frac{1}{3}\underline{c} \quad h=k,\ \frac{2}{3}h=\frac{1}{3} \quad \therefore h=k=\frac{1}{2}$

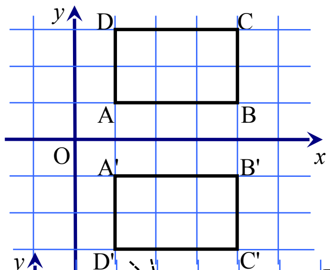
Exercise 19

1 (a)

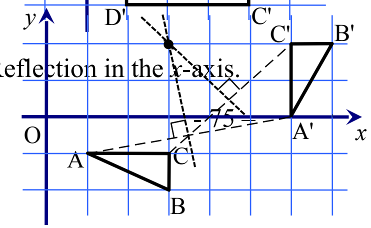


(b) Translation with column vector $\begin{pmatrix}-4\\-3\end{pmatrix}$

2 (a)



(b) Reflection in the x-axis.



3 (a)

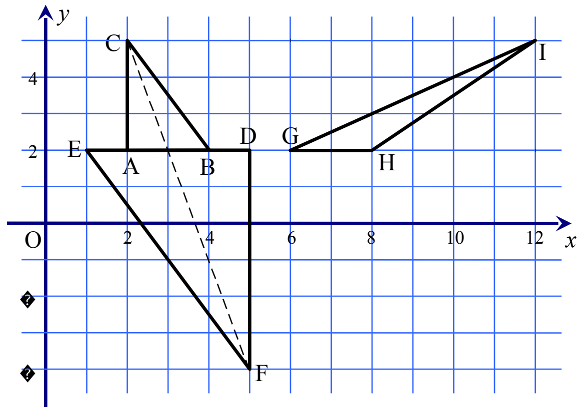
(b) (3, 2) (c) 270˚ clockwise or 90˚ anticlockwise

4 $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ 1 & 1 & 2\end{pmatrix}=\begin{pmatrix}1 & 2 & 2\\ -3 & -3 & -6\end{pmatrix}$

$a=1$, b = 0, c = 0, d = –2 $\therefore \begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}$

The scale factor is –2,

5 (a)



(b) Join A to D and C to F. The point of intersection (3, 2) is the centre of enlargement.

$\frac{\text{image length}}{\text{object length}}=\frac{DE}{AB}=\frac{-4}{2}=-2=\text{scale factor}$

∴Enlargement with centre (3, 2) and scale factor –2.

(c) The invariant line is parallel to the x-axis, so let the line be $y=a$.

A(2, 2) moves 4units and C(2, 5) moves 10units.

simsiky@gmail.com

$$\frac{4}{2-a}=\frac{10}{5-a}\quad \therefore a=0$$ The invariant line is $y=0$

i.e. x-axis. Shear factor is $\frac{4}{2-0}=2$.

$\therefore$Shear with x-axis invariant and shear factor 2.

Exercise 20

1 (a) (i) (60°N, 60°W) (ii) (30°N, 30°E)
(iii) (30°S, 30°E)
(b) (i) Difference in latitude = 30° + 30° = 60°

$$\text{Distance}=\frac{60°}{360°}\times 2\pi R=6672\text{ km}$$

(ii) Difference in longitude = 60° + 30° = 90°

$$\text{Distance}=\frac{90°}{360°}\times 2\pi R\cos 60°=5004\text{ km}$$

(c) (i) 14:00 hours

(ii) $\text{Difference}=\frac{90°}{15°}=6\text{ hours}$

14:00 – 6hours = 08:00hours

2 (a) Difference in latitude between A and B (C and D)
= 60° = 60×60′ = 3 600′
Distance AB = 3600×cos45° = 2 545. 5 nm
Distance CD = 3600×cos20° = 3 382. 8 nm
Difference in longitude between B and D (A and C)
= 25° = 25×60′ = 1500′
Distance BD = Distance AC = 1500 nm
$\therefore$Distance by jet X = 2545.5 + 1500 = 4045.5
= 4046 nm
$\therefore$Distance by jet Y =1500 + 3382.8 = 4882.8
= 4483 nm

(b)

$$\text{Time}=\frac{\text{Distance}}{\text{Speed}}=\frac{4045.5}{400}=10.1=10\text{ hours}$$

(c)

$$\text{Speed}=\frac{\text{Distance}}{\text{Time}}=\frac{4882.8}{10.1}=483=480\text{ knots}$$

Exercise 21

1 (a) $\frac{1}{6}$ (b) $\frac{3}{6}=\frac{1}{2}$ (c) $\frac{3}{6}=\frac{1}{2}$

2

$$P(\text{H and T})=P(\text{HT})+P(\text{TH})=\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{1}{2}$$

3 These events are independent.

$$P(R)=\frac{1}{6},\ P(Y)=\frac{2}{6}=\frac{1}{3},\ P(B)=\frac{3}{6}=\frac{1}{2}$$

(a) $P(YY)=P(Y)\times P(Y)=\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$

(b)

$$P(\text{R and B})=2\times P(R)\times P(B)=2\times\frac{1}{6}\times\frac{1}{2}=\frac{1}{6}$$

(c) $P(\text{no R for each selection})=1-\frac{1}{6}=\frac{5}{6}$

So for two selections, $P(\text{no R})=\frac{5}{6}\times\frac{5}{6}=\frac{25}{36}$

(d) P(two beads of the same colour)
= P(RR) + P(YY) + P(BB)

$$=\frac{1}{6}\times\frac{1}{6}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{2}=\frac{7}{18}$$

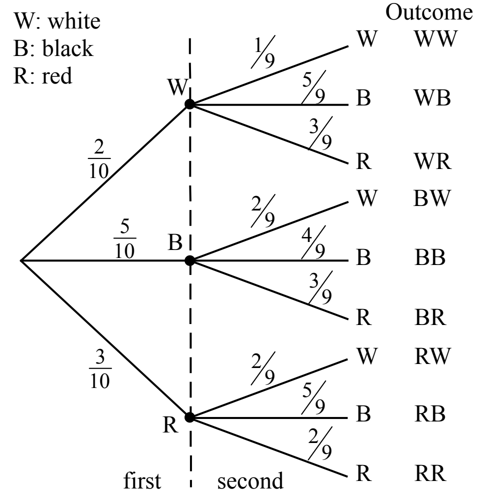
(e) P(two beads of different colours)
= 1 – P(two bead of the same colour)

$$=1-\frac{7}{18}=\frac{11}{18}$$

4 P(W from Y) = P(W from X)×P(W from Y)
+ P(B from X)×P(W from Y)

$$=\frac{2}{5}\times\frac{4}{6}+\frac{3}{5}\times\frac{3}{6}=\frac{17}{30}$$

5 (a) The tree diagram is shown below.



(b) (i) $P(RR)=\frac{3}{10}\times\frac{2}{9}=\frac{1}{15}$

(ii) P(W and B) = P(WB) + P(BW)

$$=\frac{2}{10}\times\frac{5}{9}+\frac{5}{10}\times\frac{2}{9}=\frac{2}{9}$$

(iii) P(B and R) = P(BR) + P(RB)

$$= \frac{5}{10} \times \frac{3}{9} + \frac{3}{10} \times \frac{5}{9} = \frac{1}{3}$$

(iv) P(no B) = P(WW) + P(WR) + P(RW) + P(RR)

$$= \frac{2}{10} \times \frac{1}{9} + \frac{2}{10} \times \frac{3}{9} + \frac{3}{10} \times \frac{2}{9} + \frac{3}{10} \times \frac{2}{9} = \frac{2}{9}$$

(v) P(two balls of the same colour)
= P(WW) + P(BB) + P(RR)

$$= \frac{2}{10} \times \frac{1}{9} + \frac{5}{10} \times \frac{4}{9} + \frac{3}{10} \times \frac{2}{9} = \frac{14}{45}$$

(vi) P(two balls of different colours)
= 1 – P(two balls of the same colour)

$$= 1 - \frac{14}{45} = \frac{31}{45}$$

Exercise 22

1 1day = 24hours $\frac{360°}{24} = 15°$ represents 1hour.

(a) $\frac{120°}{360°} = \frac{1}{3}$ (b) $\frac{105°}{15°} = 7$ hours (c) $2 \times 15° = 30°$

(d) Angle of sector 'eating'
= 360° – 105° – 120° – 30° – 90° = 15°

$$\frac{15}{360} \times 100 = 4.16 = 4.2\%$$

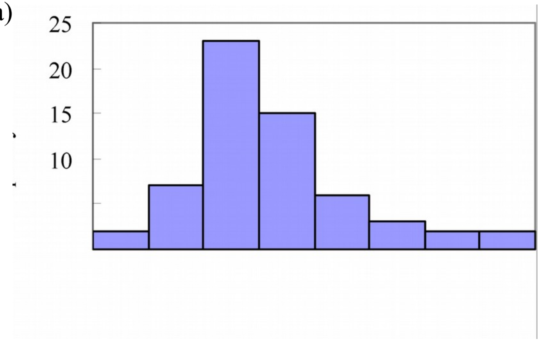
2 Arrange the numbers: 2, 3, 4, 4, 5, 6, 6, 6, 7, 8

(a)

$$\text{mean} = \frac{2+3+4+4+5+6+6+6+7+8}{10} = 5.1$$

(b) mode = 6 (c) median $= \frac{5+6}{2} = 5.5$

3 (a)



(b) Redraw the table.

Marks	0 – 9	10 – 19	20 – 29
Centre (x)	4.5	14.5	24.5
Frequency(f)	2	7	23
Product (fx)	9	101.5	563.5

Marks	30 – 39	40 – 49	50 – 59
Centre(x)	34.5	44.5	54.5
Frequency(f)	15	6	3
Product (fx)	517.5	267	163.5

Marks	60 – 69	70 – 79	Total
Centre(x)	64.5	74.5	
Frequency(f)	2	2	60
Product (fx)	129	149	1900

(i) The number of pupil = total f = 60

(ii) Mean $= \frac{\text{Sum of } fx}{\text{total } f} = \frac{1900}{60} = 31.7$ (to 1dp)

4 (a)

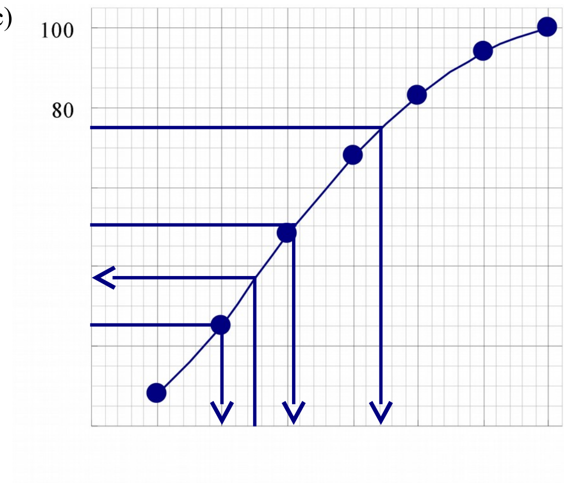
$$\text{Mean} = \frac{\text{Sum of } fx}{\text{total } f} = \frac{15 \times 8 + 25 \times 17 + \cdots + 75 \times 6}{100}$$

$$= \frac{4240}{100} = 42.4 \text{ min}$$

(b)

x	≤20	≤30	≤40	≤50	≤60	≤70	≤80
f	8	25	48	68	83	94	100

(c)



(d) (i) The median = 41 minutes
(ii) The interquartile range = 54 – 30 = 24 minutes

(e) The frequency at 35minutes is 37.

$$\text{The probability} = \frac{37}{100}$$

Exercise 23

1 (a) 11 + 3 = 14 and 14 + 3 = 17
(b) 2 – 2 = 0 and 0 – 2 = –2
(c) 100 × 10 = 1 000 and 1 000 × 10 = 10 000

simsiky@gmail.com

(d) $-2\times\left(-\frac{1}{2}\right)=1$ and $1\times\left(-\frac{1}{2}\right)=-\frac{1}{2}$

(e) 11 + 5 = 16 and 16 + 6 = 22

(f) $6^2 = 36$ and $7^2 = 49$

2 (a) $a = 1$, $d = 2$ $\therefore a_n = 1+(n-1)\times 2 = 2n-1$

(b) $a = 10$, $d = -4$

$\therefore a_n = 10+(n-1)\times(-4) = -4n+6$

(c) $a = 2$, $r = 3$ $\therefore a_n = 2\times 3^{n-1}$

(d) $a_n = (n-1)^2$ (e) $a_n = n^3$

3 Sequence: 5, 8, 11, . . .

(a) Each term is 3 more than the previous term.

∴4th term = 11 + 3 =14

(b) first term(a) =5, common difference (d) = 3

$\therefore a_n = 5+(n-1)\times 3 = 3n+2$

(c) $44 = 3n + 2$ $\therefore n = 14$

(d) $a_n = 5+(25-1)\times 3 = 77$